MAT 22B: PROBLEM SET 3

DUE TO FRIDAY OCT 18 AT 10:00AM

ABSTRACT. This is the third problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Oct 18 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice the basic techniques for solving second-order differential equations. In particular, we would like to become familiar with characteristic equations, damped oscillations and the method of undetermined coefficients.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following differential equations, find all solutions:

(a) $y''(t) = 0$,

There are different ways to do this problem. The easiest is to take the anti-derivative twice getting the solution

$$y(t) = c_1 t + c_2, \qquad c_1, c_2 \in \mathbb{R}.$$

(b) $y''(t) = t^3 - 3t^2 + 7$,

For this problem, integrating twice resulting with two extra constants resulting in

$$y(t) = \frac{1}{20}t^5 - \frac{1}{4}t^4 + \frac{7}{2}t^2 + c_1 t + c_2, \quad c_1, c_2 \in \mathbb{R}$$

(c) $y''(t) = 144y(t)$,

For this problem, we can find the roots getting the characteristic polynomial

$$\lambda^2 - 144 = 0$$

and then we get two roots resulting in the general solution

$$y(t) = c_1 e^{12t} + c_2 e^{-12t}, \quad c_1, c_2 \in \mathbb{R}.$$

(d) $y''(t) = -9y(t)$.

Let us use the characteristic polynomial once again, which is

$$\lambda^2 + 9.$$

This results in two complex roots $\pm 3i$, which after rewriting by using Euler's formula read:

$$y(t) = C_1 e^{3it} + C_2 e^{-3i} = C_3 \cos(3t) + C_4 \sin(3t),$$

where the constants are complex numbers.

Problem 2. Consider the following differential equation:

$$y''(t) = 5y'(t).$$

(a) Find all solutions $y(t)$ to the differential equation above.

Let $y(t) = e^{\lambda t}$ then plug into the formula.

$$(\lambda^2 - 5\lambda)e^{\lambda t} = 0$$

now you have two distinct solutions with the characteristic polynomial. Solve for lambda and you find $\lambda = 0, 5$ so for the general solutions you get

$$y = c_1 + c_2 e^{5t}$$

(b) Find two distinct solutions $y_1(t)$ and $y_2(t)$ such that $y_1(0) = 0$ and $y_2(0) = 0$.

The constant function $y_1(t) = 0$ satisfies this. The solution $y_2(t) = 1 - e^{5t}$ also satisfies $y_2(0) = 0$.

(c) How many solutions are there such that $y(0) = 0$ and $y'(0) = 1$?

Let use the initial conditions with general solution s.t

$$0 = c_1 + c_2$$

and when you take the derivative we get

$$y'(t) = 5c_2 e^{5t}$$

so using the initial conditions we get that $c_2 = \frac{1}{5}$ so $c_1 = -\frac{1}{5}$ since the initial conditions exist that means that their is only one solution which is

$$y(t) = -\frac{1}{5} + \frac{1}{5}e^{5t}$$

Problem 3. (20 pts) Consider the following second-order differential equation:

$$y''(t) + 3y'(t) = -2y(t).$$

(a) Suppose that $y(t) = e^{\lambda t}$ is a solution to the differential equation above. Find all possible values of $\lambda \in \mathbb{C}$.

Let us set up the equation in standard form i.e.

$$y''(t) + 3y'(t) + 2y(t) = 0.$$

Given the exponential ansatz $y(t) = e^{\lambda t}$, we can plug in the equation and get

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0$$

which can be separated to

$$(\lambda^2 + 3\lambda + 2)e^{\lambda t} = 0$$

This leads to the characteristic equation $\lambda^2 + 3\lambda + 2 = 0$, which we then solve for λ, yielding $\lambda_1 = -1$ and $\lambda_2 = -2$.

(b) Find two linearly independent solutions of the differential equation above. In this context, two solutions $y_1(t), y_2(t)$ are linearly independent if $y_1(t)$ is not a constant multiple of $y_2(t)$.

The two solutions that are linearly independent are $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. To prove that these are linearly independent is taking the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -2e^{-3t} + e^{-3t} = -e^{-3t}$$

since the Wronskian is not equal to zero, these solutions are independent.

(c) Use superposition to find *all* solutions to the differential equation.

The law superposition states that adding two valid solutions will produce a third i.e.

$$y_3 = c_1 y_1 + c_2 y_2$$

is will be a solution. By the existence and uniqueness theorem for second-order differential equations, *all* solutions are obtained this way. In the case of this problem, we can use this law to find all possible solutions:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}.$$

(d) Find the unique solution $y_1(t)$ such that $y_1(0) = 0$ and $y_1'(0) = 5$.

To solve for initial conditions problems of 2nd degree polynomials we will have two equations and two unknowns so well start with the first condition and get

$$0 = c_1 + c_2$$

and let $c_2 = -c_1$ then for the second condition we get

$$5 = -c_1 - 2c_2$$

then using the substitution from above we get that $c_1 = 5$ and then $c_2 = -5$ so the solution would be

$$y(t) = 5e^{-t} - 5e^{-2t}.$$

(e) Describe the long-term behavior of *all* solutions to the differential equation.

We can take the general form for solutions to the differential equation. We will see that if we take the limit as time goes to infinity that these solutions all go to zero. Indeed, the limit is:

$$\lim_{t\to\infty} c_1\frac{1}{e^t} + c_2\frac{1}{e^{2t}} = 0.$$

Problem 4. (20 pts) Consider the following differential equation:

$$y''(t) - 6y'(t) + 10y(t) = 0.$$

(a) Find the characteristic equation associated to the differential equation above.

To find the characteristic equation use $y(t) = e^{\lambda t}$ such that

$$\lambda^2 e^{\lambda t} - 6\lambda e^{\lambda t} + 10e^{\lambda t}$$

we get the characteristic equation

$$\lambda^2 - 6\lambda + 10 = 0.$$

(b) Find *all* solutions to the differential equation above.

To find the solutions of the characteristic polynomial equal it to zero and solve for λ s.t you should get

$$\lambda_1 = \frac{6 + \sqrt{36 - 40}}{2} = 3 + i$$

and

$$\lambda_2 = \frac{6 - 1\sqrt{36 - 40}}{2} = 3 - i$$

so by using we get that the solution of all solutions

$$y(t) = c_1 e^{(3+i)t} + c_2 e^{(3-i)t}.$$

Now, using the Euler's formula, as explained in class, the solution also reads

$$y(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t),$$

where c_1, c_2 are complex numbers.

(c) Is there any constant solution $y_1(t)$?

A constant solution must have $y_1'(t) = 0$ and $y_1''(t) =$. Thus, a constant solution must satisfy $10y_1(t) = 0$ and hence the only constant solution is $y_1(t) = 0$.

(d) Find the long-term behaviour

$$\lim_{t\to\infty} y(t)$$

for all solutions $y(t)$ to the differential equation.

The constant solution $y_1(t)$ will converge to 0. All other solutions will oscillate, since they are a superposition of sines and cosine, thus their long-term behavior does *not* exist.

(e) Find the unique solution $y_2(t)$ satisfying $y_2(0) = 2$ and $y_2'(0) = 9$ and plot its graph. How many zeroes does it have ?

First, we will use the initial condition $y_2(0) = 2$ such that

$$2 = c_1 e^0 cos(0) + c_2 e^0(0).$$

We solve for the constant to get $c_1 = 2$. Now, let us take the derivative of $y(t)$ and use the second initial condition such that

$$y' = -3c_1 e^{3t} \sin(t) + 3c_2 e^{3t} \cos(t).$$

Plugging in the initial condition we get

$$9 = 0 + 3c_2.$$

Solving for the constant we get $c_2 = 3$, and thus the solution to the initial problem values would be

$$y(t) = 2e^{3t} \cos(t) + 3e^{3t} \sin(t).$$

Problem 5. (20 pts) Let $\gamma \in \mathbb{R}$ be a positive real number and consider the damped system modeled by the following second-order differential equation:

$$y''(t) + \gamma y'(t) + 25y(t) = 0,$$

(a) Show that the long-term behaviour of *all* solutions is independent of γ.

The characteristic equation is

$$\lambda^2 + \gamma\lambda + 25 = 0.$$

The roots will be of the form

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4(1)(25)}}{2}.$$

There are three cases, depending on $\gamma^2 - 100$ being positive, zero or negative. If $\gamma^2 - 100 < 0$, then the solutions are of the form $e^{-\gamma t/2}$ times sines and cosines. Thus their long-term behavior is zero, since γ is positive. If $\gamma^2 - 100 = 0$, then we obtain a repeated root and solutions are of the form $c_1 e^{-5t} + c_2 t e^{-5t}$, which converges to zero as t goes to infinity. Finally, for $\gamma^2 - 100 > 0$, we observe that $-\gamma$ is strictly greater than $\sqrt{\gamma^2 - 100}$, and thus the general solution is a combination of exponentials with negative exponents. Hence, their long-term behavior is also zero.

(b) For which values of $\gamma \in \mathbb{R}^+$ does the above differential equation have oscillating solutions ? (i.e. solutions with infinitely many zeroes.)

Oscillating solutions correspond to complex roots for the characteristic equation, thus we must have

$$\gamma^2 - 100 < 0.$$

Since we are only using positive solutions for γ, we get that the following conditions for oscillating solutions

$$0 < \gamma < 10.$$

(c) Classify the above damped system into underdamped, critically damped and overdamped in terms of γ, and qualitatively draw solutions corresponding to each of these three cases.

For $0 < \gamma < 10$, the system is underdamped. For $\gamma = 10$, the system is critically damped, and for $10 < \gamma$ the system is overdamped.

(d) Find all solutions to the above differential equation for $\gamma = 10$.

Let us solve by using the characteristic equation:

$$\lambda^2 + 10\lambda + 25 = 0.$$

We get that both roots will be $\lambda = -5$. Since it is a repeated root, two linearly independent solutions will be e^{-5t} and te^{-5t}. Thus the general solution is

$$y(t) = c_1 e^{-5t} + c_2 t e^{-5t}, \qquad c_1, c_2 \in \mathbb{R}.$$

(e) For $\gamma = 10$, let $y_1(t)$ be a solution such that $y_1(0) = 0$ and $y(1) = 1$, and $y_2(t)$ a solution such that $y_2(0) = 1$ and $y_2'(0) = 0$. Compute the ration $y_1(t)/y_2(t)$ for $t \to \infty$ and deduce which solution converges faster to zero.

By substituting the corresponding initial conditions we obtain that

$$y_1(t) = e^{5-5t} t, \qquad y_2(t) = e^{-5t}(1 + 5t).$$

The quotient thus converges to $\frac{1}{5}e^5$, which is larger than 1. Thus $y_1(t) > y_2(t)$ in the limit, and $y_2(t)$ converges faster to zero.

Problem 6. (20 pts) Let us consider the following damped system with external forced vibrations:

$$y''(t) + 2y'(t) - 35y(t) = \cos(4t).$$

(a) Find *one* solution to the above differential equation.

The method of underdetermined coefficients states that the ansatz is of the form $A\sin(4t) + B\cos(4t)$. By inserting into the differential equation we obtain:

$$A = \frac{8}{2665}, \quad B = \frac{-51}{2665}.$$

Thus a particular solution is

$$y(t) = \frac{8}{2665}\sin(4t) - \frac{51}{2665}\cos(4t).$$

(b) Find *all* solutions to the differential equation by using the principle of superposition. Plot them qualitatively.

The general solution for the homogeneous problem is

$$y(t) = e^{-7t} + e^{5t.}$$

Since this problem is linear non-homogeneous, the general solution is obtained by adding a particular solution to this homogeneous general solution. The general solution is thus

$$y(t) = c_1e^{-7t} + c_2e^{5t} + \frac{8}{2665}\sin(4t) - \frac{51}{2665}\cos(4t).$$

(c) Does there exist a solution $y_1(t)$ whose long-term behavior is infinite ?

Yes, any solution with $c_2 \neq 0$ will converge to infinity in the long-term.

(d) Let us increase our oscillating forced vibration from $\cos(4t)$ to $\cos(4t) + 9e^{6t}$. This yields the system

$$y''(t) + 2y'(t) - 35y(t) = \cos(4t) + 9e^{6t}.$$

Find all solutions to this differential equation.

By superposition, it suffices to find a particular solution for the differential equation:

$$y''(t) + 2y'(t) - 35y(t) = 9e^{6t}.$$

The method of undetermined coefficients yields the guess $y(t) = Ae^{6t}$, with coefficient $A = \frac{9}{13}$. Thus, by the principle of superposition, the general solution is

$$y(t) = c_1e^{-7t} + c_2e^{5t} + \frac{8}{2665}\sin(4t) - \frac{51}{2665}\cos(4t) + \frac{9}{13}e^{6t}.$$

Problem 7. (20 pts) Consider the following second-order differential equation:

$$y''(t) - 6y'(t) + 10y(t) = te^{5t},$$

(a) Find a particular solution $y_p(t)$ to the differential equation above.

To find a particular solution you will have to use the method of undetermined coefficients such that you guess that a solution is of form

$$y(t) = Ate^{5t} + Be^{5t}$$

from here you will solve for the coefficients using the first and second derivatives s.t $A = \frac{1}{5}$and $B = -\frac{4}{25}$ then the practical solution for this problem is

$$y_p = e^{5t}(\frac{t}{5} - \frac{4}{25})$$

(b) Find the general solution to the differential equation above.

To find the general solution we just have to find the homogeneous solution since we found the particular solution so we solve using the characteristc polynomial such that

$$\lambda^2 - 6\lambda + 10 = 0$$

from here we will get two complex roots such that $\lambda_1 = 3 + i$and $\lambda_2 = 3 - i$ from the complex root identity we get the solution

$$y(t) = e^{3t}(c_1 \cos(t) + c_2 \sin(t))$$

as the general solution to find the total solution we have to use the law of superposition and get

$$y(t) = e^{3t}(c_1 \cos(t) + c_2 \sin(t)) + e^{5t}(\frac{t}{5} - \frac{4}{25})$$

(c) How many solutions $y(t)$ are there with $y(0) = 0$ and $y'(0) = 1$? If any, find them and qualitatively plot their graphs.

Let us consider the general solution

$$y(t) = e^{3t}(c_1 \cos(t) + c_2 \sin(t)) + e^{5t}(\frac{t}{5} - \frac{4}{25})$$

and also the derivative

$$y'(t) = 3e^{3t}(c_1 \cos(t) + c_2 \sin(t)) + e^{3t}(-c_1 \sin(t) + c_2 \cos(t) + 5e^{5t}(\frac{t}{5} - \frac{4}{25}) + e^{5t}(\frac{1}{5}).$$

By imposing the initial conditions we obtain $c_1 = \frac{4}{25}$ and $c_2 = \frac{28}{25}$. Thus there is a unique solution for those initial conditions. Its graph is qualitatively the same graph as for e^{5t}.

(d) Let us decide to approximate $e^{5t} \approx 1 + 5t + \frac{125}{6}t^2 + O(t^3)$ via a Taylor expansion and consider the system:

$$x''(t) - 6x'(t) + 10x(t) = t(1 + 5t + \frac{125}{6}t^2).$$

Find a particular solution $x_p(t)$ to this differential equation.

Let us use the method of undetermined coefficients, where we would use the the function

$$x_p = A + Bt + Ct^2 + Dt^3.$$

By inserting inside the differential equation and solving we obtain

$$x_p = \frac{25}{12}t^3 + \frac{17}{4}t^2 + \frac{79}{20}t + \frac{38}{25}.$$

(e) Compare the solution $y_p(t)$ obtained in Part (a) with the particular solution $x_p(t)$. Is one related to the Taylor expansion of the other ? Do their long-term behaviors coincide ?

The Taylor expansion of the particular solution in Part (a) has essentially nothing to do with the particular solution $x_p(t)$. Thus, comparing Taylor expansions to particular solutions yields no similarities. The long-term behaviors coincide, but that is a coincidence. That said, had we compared solutions to the differential equations *given* the same set of initial conditions, then we would have obtained that the solutions $x(t)$ are Taylor expansions to the solutions $y(t)$. Either answer, if properly discussed, will have a full grade.