MAT 22B: PROBLEM SET 4

DUE TO FRIDAY NOV 8 AT 10:00AM

Abstract. This is the fourth problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Nov 8 at 10:00am via online submission.

Purpose: The goal of this assignment is to practice the Laplace transform. In particular, we would like to become familiar with solving differential equations and Initial Value Problems by using Laplace transforms.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following functions $f(t), g(s)$, find the Laplace transform $\mathcal{L}(f(t))$ or the anti-Laplace transform $\mathcal{L}(g(s))$ correspondingly.

(a) $\mathcal{L}(3)$, $\mathcal{L}(e^{3t}\sin(2t))$, $\mathcal{L}(e^{4t}\cos(6t))$, $\mathcal{L}(t^3\sin(4t))$, $\mathcal{L}(t^2 + t\cosh(5t))$,

(b) $\mathcal{L}^{-1}\left(\dfrac{2+s}{s^3}\right)$, $\mathcal{L}^{-1}\left(\dfrac{3}{s^2-7}\right)$, $\mathcal{L}^{-1}\left(\dfrac{12s}{s^2+1}\right)$, $\mathcal{L}^{-1}\left(\dfrac{1}{(s-1)^2+2}\right)$, $\mathcal{L}^{-1}\left(\dfrac{s-1}{(s-1)^2+3}\right)$.

Problem 2. Show that the Laplace transform satisfies the following properties:

(a) $\mathcal{L}(e^{ct} \cdot f(t))(s) = \mathcal{L}(f(t))(s-c)$, where $c \in \mathbb{R}$.

(b) $\mathcal{L}(u_c(t) \cdot f(t-c))(s) = e^{-cs} \cdot \mathcal{L}(f(t))(s)$, where $c \in \mathbb{R}$.

(c) $\mathcal{L}(t^n \cdot f(t))(s) = (-1)^n \cdot \mathcal{L}(f(t))^{(n)}(s)$.

Problem 3. (20 pts) Solve the following four Initial Value Problems using the method of the Laplace transform:

(a) $y''(t) + 9y(t) = e^{4t}, \quad y(0) = 1, \quad y'(0) = 0,$

(b) $y''(t) + 6y'(t) + 6y(t) = \sin(2t)$, $y(0) = 0$, $y'(0) = 1$.

(c) $y^{(4)}(t) - 4y(t) = 0, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$

(d) $y^{(10)}(t) - 15y^{(8)}(t) + 85y^{(6)}(t) - 225y^{(4)}(t) + 274y^{(2)}(t) - 120y(t) = 0,$

with the following set of initial conditions

$$\begin{gathered} y(0) = 0, \quad y'(0) = 0, \quad y^{(2)}(0) = 0, \quad y^{(3)}(0) = 0, \quad y^{(4)}(0) = 0, \\ y^{(5)}(0) = 0, \quad y^{(6)}(0) = 0, \quad y^{(7)}(0) = 0, \quad y^{(8)}(0) = 0, \quad y^{(9)}(0) = 1. \end{gathered}$$

Hint: The polynomial

$$s^{10} - 15s^8(t) + 85s^6 - 225s^4 + 274s^2 - 120$$

factorizes as $(s^2-1)(s^2-2)(s^2-3)(s^2-4)(s^2-5)$.

Problem 4. (20 pts) Consider the following Initial Value Problem:

$$y''(t) + ty'(t) - 2y(t) = 2, \quad y(0) = 0, y'(0) = 0.$$

This is a non-homogeneous linear second-order differential equation with *non-constant* coefficients and *not* of Euler type. Note that we have not seen a previous method to solve this, and thus Laplace transform is essentially the only method at this point.

(a) Write the Laplace transform of the Initial Value Problem above.

(b) Find a closed formula for the Laplace transform $\mathcal{L}(y(t))$.

Hint: You will have to solve a first-order differential equation on $\mathcal{L}(y(t))$.

(c) Find the unique solution $y(t)$ to the Initial Value Problem.

Problem 5. (20 pts) Let $u_c(t)$ be the Heaviside step function at $c \in \mathbb{R}$. In this problem we study a *third-order differential equation* with a discontinuous external force. Solve the following initial value problem:

$$y'''(t) - 4y''(t) = 4t + 3u_6(t)e^{30-5t}, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 4.$$

Problem 6. (20 pts) Let $\delta(t)$ be the Dirac Delta distribution. In this problem we will study pendulums at rest which are suddenly affected by an impulse.

(a) Let us consider a pendulum of angular frequency $\omega = 2$ with no friction at small oscillations, modeled by $y''(t) + 4y(t) = 0$. Find the unique solution to the following Initial Value Problem:

$$y''(t) + 4y(t) = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Plot the solution $y(t)$ you have obtained. Note that the initial conditions have the pendulum at absolute rest before the impulse at $t = \pi$ is exerted.

(b) How do the solutions of the Initial Value Problem

$$y''(t) + 4y(t) = \gamma \cdot \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 0$$

vary for different values of $\gamma \in \mathbb{R}$? In particular, what are the differences between positive and negative values of γ ?

Qualitatively plot the solutions for $\gamma = -10, -5, 0, 5, 10$.

(c) How do the solutions of the Initial Value Problem

$$y''(t) + 4y(t) = \delta(t - c), \quad y(0) = 0, \quad y'(0) = 0$$

differ from each other for different values of $c \in \mathbb{R}^+$?

Qualitatively plot the solutions for $c = 1, 6, 12$.

(d) Find the unique solution to the following Initial Value Problem:

$$y''(t) + 4y(t) = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Plot the solution $y(t)$ you have obtained. Note that the initial conditions have the pendulum at absolute rest before the two impulses, at $t = \pi$ and $t = 2\pi$ are exerted.

(e) Let us consider instead an *inverted* pendulum with no friction, modeled by

$$y''(t) - y(t) = 0.$$

Let us study the effect of inserting an impulse after three seconds. Find the unique solution to the following Initial Value Problem:

$$y''(t) - y(t) = -20 \cdot \delta(t - 3), \quad y(0) = 2, \quad y'(0) = 4.$$

Problem 7. (20 pts) Consider a horizontal steel beam $B = [0, 2]$ of length 2 meters, such that its elastic modulus and second area momentum are normalized at 1. In this problem we will study the *deflection* of a beam when a force $F(x)$ is applied at a point $x \in B$ in the beam. The Euler-Bernoulli theory of beams states that the deflection $y(x)$ at point $x \in B = [0, 2]$ given by a force $F(x)$ is modeled by the following fourth-order differential equation:

$$y^{(4)}(x) = F(x).$$

with boundary conditions $y(0) = 0$, $y(2) = 0$, which state that there is no deflection at the endpoints, and also $y''(0) = 0$, $y''(2) = 0$. The goal of this problem is to understand how the beam will bend if we hit the beam at point $c \in [0, 2]$ with an impulse force

$$F(x) = \delta(x - c).$$

Note that this is the differential equation that was used to construct several large metal structures in the 19th century, including the Eiffel Tower (Paris 1889) and the first Ferris wheels (e.g. Chicago 1893).

(a) Solve the initial value problem:

$$y^{(4)}(x) = -\delta(x - 1),$$
$$y(0) = 0, \quad y(2) = 0, \quad y''(0) = 0, \quad y''(2) = 0.$$

(b) How much does the steel beam deflect at its center $x = 1$ if we apply the impulse $-\delta(x - 1)$ as in the above model ?

(c) How much does the steel beam deflect at its center $x = 1$ if we apply the two impulses $-\delta(x-1)$ and $-\delta(x-1.5)$ at the center and at three-halves of the beam ?