MAT 22B: PROBLEM SET 5

DUE TO FRIDAY NOV 15 AT 11:59PM

Abstract. This is the fifth problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Nov 15 at 11:59pm via submission to Gradescope.

Purpose: The goal of this assignment is to practice solving linear systems of differential equations. In particular, we would like to become familiar with solving linear systems of differential equations with constant coefficients by using eigenvalues and eigenvectors.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following matrices, find their eigenvalues and, for each eigenvalue, find at least one eigenvector.

$$\begin{pmatrix} 4 & -2 \\ -3 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & -8 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 7 & 4 & 6 \\ -3 & -1 & -8 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Problem 2. Find all constant solutions of the following linear system of differential equations with constant coefficients:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & -5 \\ 2 & 4 & -10 \\ -1 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Problem 3. (20 pts) Consider the following linear system of differential equations:

$$x_1'(t) = 3x_1(t) - 6x_2(t) + 12x_3(t), \quad x_2'(t) = 2x_2(t) - 6x_3(t), \quad x_3'(t) = -x_3(t).$$

(a) Give a fundamental set of solutions for the above system of differential equations. (You must prove linear independence.)

(b) Find *all* solutions $(x_1(t), x_2(t), x_3(t))$ to the system of differential equations.

(c) Find the unique solution such that

$$x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = -1.$$

Problem 4. (20 pts) Consider the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

(a) Provide a fundamental set of solutions.

(b) Find the unique solution such that

$$x_1(4) = 1, \quad x_2(4) = 6, \quad x_3(4) = -13.$$

(c) What are all constant solutions to the system ?

(d) Compute the long-term behavior of the unique solution such that

$$x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = 0.$$

Problem 5. (20 pts) Consider the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

(a) Draw the direction field associated to the above systems of differential equations in the (x_1, x_2)-plane. Is there any constant solution ?

(b) Let $(x_1(t), x_2(t))$ be the unique solution such that $(x_1(0), x_2(0)) = (3, 4)$. Find the norm of the vector $(x_1(15), x_2(15))$ at time $t = 15$.

(c) Find a set of fundamental solutions for the system.

(d) Let $a, b \in \mathbb{R}$ be two real numbers and $(x_1(t), x_2(t))$ the unique solution such that $(x_1(0), x_2(0)) = (a, b)$. Describe the motion of the vector $(x_1(t), x_2(t))$ in the (x_1, x_2)-plane as t ranges from $t = 0$ to $t \to \infty$.

Remark: The initial matrix for this system in Problem 5 was given by:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

You are welcome to solve this Problem 5 with either of the two matrices. The one in the statement is easier and will lead to a neater solution. I will in addition explain the solution in class on Friday Nov 15. That said, if you have already worked through the problem with the system above, this is also good.

Problem 6. (20 pts) In this problem we explore the different type of linear systems of differential equations with repeated eigenvalues.

(a) Provide a fundamental set of solutions for the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

(b) Find a fundamental set of solutions for the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 10 & -2 \\ 18 & -2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

(c) Find the unique solution to the above 2×2 system which satisfies

$$(x_1(0), x_2(0)) = (1, 0).$$

Problem 7. (20 pts) Consider the following non-homogeneous system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} -1 \\ -6 \\ -5 \end{pmatrix}$$

(a) Find all solutions to the above linear non-homogeneous system.

(b) Find the unique solution such that

$$(x_1(0), x_2(0), x_3(0)) = (0, 0, 0).$$