MAT 22B: PROBLEM SET 5

DUE TO FRIDAY NOV 15 AT 11:59PM

Abstract. This is the fifth problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Nov 15 at 11:59pm via submission to Gradescope.

Purpose: The goal of this assignment is to practice solving linear systems of differential equations. In particular, we would like to become familiar with solving linear systems of differential equations with constant coefficients by using eigenvalues and eigenvectors.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following matrices, find their eigenvalues and, for each eigenvalue, find at least one eigenvector.

$$\begin{pmatrix} 4 & -2 \\ -3 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & -8 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 7 & 4 & 6 \\ -3 & -1 & -8 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Solution: Method: to find eigenvalues of a matrix A, first we solve an algebraic equation for λ:

$$\det(A - \lambda I) = 0$$

Then we solve a linear system in vector v:

$$(A - \lambda I)v = 0,$$

using λ obtained from the first step. Note that this is always a degenerate system, meaning that there is at least one free variable in v for which you solve! The number of free variables is the number of linearly independent eigenvectors associated with the eigenvalue λ.

Finally, choose arbitrary number for the free variables, but keep linear independence if there are multiple of them, to get the eigenvector!

Results: denote the matrices above by $A_1, A_2, \ldots, A_6$ from top left to bottom right. We list their eigenvalues and eigenvectors below: for A_1

$$\begin{array}{ll} \lambda_1(A_1) = 3 & \lambda_2(A_1) = 10 \\ v_1(A_1) = [2\ \ 1]^\top & v_2(A_1) = [-1/3\ \ 1]^\top. \end{array}$$

For A_2,

$$\begin{array}{ll} \lambda_1(A_2) = -8 & \lambda_2(A_2) = 1 \\ v_1(A_2) = [-1/3\ \ 1]^\top & v_2(A_2) = [1\ \ 0]^\top. \end{array}$$

For A_3,

$$\begin{array}{ll} \lambda_1(A_3) = i & \lambda_2(A_3) = -i \\ v_1(A_3) = [-i\ \ 1]^\top & v_2(A_3) = [i\ \ 1]^\top. \end{array}$$

For A_4,

$$\begin{array}{ll} \lambda_1(A_4) = \lambda_2(A_4) = 1 & \lambda_3(A_4) = 5 \\ v_1(A_4) = [-2\ \ 3\ \ 0]^\top & v_3(A_4) = [-2\ \ 1\ \ 0]^\top. \end{array}$$

For A_5,

$$\begin{array}{ll} \lambda_1(A_5) = 1 & \lambda_2(A_5) = \lambda_3(A_5) = 4 \\ v_1(A_5) = [0\ \ 1\ \ 0]^\top & v_2(A_5) = [0\ \ 0\ \ 1]^\top. \end{array}$$

For A_6,

$$\begin{array}{lll} \lambda_1(A_6) = 1 & \lambda_2(A_6) = i & \lambda_3(A_6) = -i \\ v_1(A_6) = [0\ \ 1\ \ 0]^\top & v_2(A_6) = [-i\ \ 0\ \ 1]^\top & v_3(A_6) = [i\ \ 0\ \ 1]^\top. \end{array}$$

Remark: these answers may differ in order or by a constant factor in eigenvectors from yours.

Problem 2. Find all constant solutions of the following linear system of differential equations with constant coefficients:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & -5 \\ 2 & 4 & -10 \\ -1 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Solution: when x is a constant solution we have $x' = 0$. Therefore the problem turns in to solving a linear system: denote the matrix above by A, then we have:

$$Ax = 0.$$

The RREF of A is just:

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore we choose a basis of the nullspace:

$$x_1 = [2 \;\; 1 \;\; 0]^\top \quad x_2 = [-5 \;\; 0 \;\; 1]^\top$$

and any constant solution x can be written as:

$$x = c_1 x_1 + c_2 x_2,$$

where c_1, c_2 are arbitrary constants.

Problem 3. (20 pts) Consider the following linear system of differential equations:

$$x_1'(t) = 3x_1(t) - 6x_2(t) + 12x_3(t), \quad x_2'(t) = 2x_2(t) - 6x_3(t), \quad x_3'(t) = -x_3(t).$$

(a) Give a fundamental set of solutions for the above system of differential equations. (You must prove linear independence.)

(b) Find *all* solutions $(x_1(t), x_2(t), x_3(t))$ to the system of differential equations.

(c) Find the unique solution such that

$$x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = -1.$$

Solution:

(a) First we write the matrix in the equation, denoted by A:

$$A = \begin{bmatrix} 3 & -6 & 12 \\ 0 & 2 & -6 \\ 0 & 0 & -1 \end{bmatrix}.$$

The eigenvalues and eigenvectors are:

$$\begin{matrix} \lambda_1 = -1 & \lambda_2 = 2 & \lambda_3 = 3 \\ v_1 = [0 \;\; 2 \;\; 1]^\top & v_2 = [6 \;\; 0 \;\; 1]^\top & v_3 = [1 \;\; 0 \;\; 0]^\top \end{matrix}.$$

Linear independence comes from the fact that all eigenvalues are distinct therefore eigenvectors associated to different eigenvalues are linearly independent. The fundamental set of solutions is:

$$S = \{v_1 e^{-1t}, v_2 e^{2t}, v_3 e^{3t}\}.$$

Or, by computing the Wronskian of S, one can also get linear independence.

(b) The general solution is just the linear combination of elements in S:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t}.$$

(c) We only need to find the corresponding constants c_1, c_2, c_3 to get this unique solution. Plug-in the initial condition and we get a linear system in c_1, c_2, c_3:

$$\begin{bmatrix} 0 & 6 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Solving this linear system gives $c_1 = 0, c_2 = -1, c_3 = 7$.

Problem 4. (20 pts) Consider the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

(a) Provide a fundamental set of solutions.

(b) Find the unique solution such that

$$x_1(4) = 1, \quad x_2(4) = 6, \quad x_3(4) = -13.$$

(c) What are all constant solutions to the system ?

(d) Compute the long-term behavior of the unique solution such that

$$x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = 0.$$

Solution:

(a) The eigenvalues and eigenvectors are:

$$\begin{array}{lll} \lambda_1 = 0 & \lambda_2 = -4 & \lambda_3 = 3 \\ v_1 = [-1/13 \ \ -6/13 \ \ 1]^\top & v_2 = [-1 \ \ 2 \ \ 1]^\top & v_3 = [-1 \ \ -3/2 \ \ 1]^\top, \end{array}$$

hence the fundamental set is:

$$S = \{v_1, v_2 e^{-4t}, v_3 e^{3t}\}.$$

(b) The constant c_1, c_2, c_3 should satisfy:

$$\begin{bmatrix} -\frac{1}{13} & -e^{-16} & -e^{12} \\ -\frac{6}{13} & 2e^{-16} & -\frac{3}{2}e^{12} \\ 1 & e^{-16} & e^{12} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}.$$

Note that the RHS is just a multiple of the first column. Then we get $c_1 = -13, c_2 = 0, c_3 = 0$.

(c) The constant solution corresponds to the zero eigenvalue, meaning that every multiple of v_1 is a constant solution.

(d) Plug-in the initial condition and we get $c_1 = 13/12, c_2 = -9/28, c_3 = -16/21$. Set $t \to \infty$, then the e^{4t} term vanishes and

$$c_3 v_3 e^{3t} = -\frac{16}{21}\begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} e^{3t} \to \begin{bmatrix} +\infty \\ +\infty \\ -\infty \end{bmatrix}.$$

While the first term is constant hence neglected, the long term behavior is $[+\infty, +\infty, -\infty]^\top$.

Problem 5. (20 pts) Consider the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

(a) Draw the direction field associated to the above systems of differential equations in the (x_1, x_2)-plane. Is there any constant solution ?

(b) Let $(x_1(t), x_2(t))$ be the unique solution such that $(x_1(0), x_2(0)) = (3, 4)$. Find the norm of the vector $(x_1(15), x_2(15))$ at time $t = 15$.

(c) Find a set of fundamental solutions for the system.

(d) Let $a, b \in \mathbb{R}$ be two real numbers and $(x_1(t), x_2(t))$ the unique solution such that $(x_1(0), x_2(0)) = (a, b)$. Describe the motion of the vector $(x_1(t), x_2(t))$ in the (x_1, x_2)-plane as t ranges from $t = 0$ to $t \to \infty$.

Solution:

(b) We can first solve part (c) then go back to this. Using the fundamental set S, the general solution is:

$$x = c_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{-t}.$$

Applying the initial condition $(x_1(0), x_2(0)) = (3, 4)$ gives

$$c_1 = 1 + \tfrac{3\sqrt{3}}{4} \quad c_2 = \sqrt{3} - \tfrac{3}{4}.$$

And the square norm of x is given by:

$$\|x\|^2 = x^\top x = 4c_1^2 e^{2t} + 4c_2^2 e^{-2t}.$$

Plug in $t = 15$ and c_1, c_2, then we have

$$\|x(15)\| = \sqrt{4\left[\left(1 + \frac{3\sqrt{3}}{4}\right)^2 e^{30} + \left(\sqrt{3} - \frac{3}{4}\right)^2 e^{-30}\right]}$$

(c) The eigenvalues and eigenvectors are:

$$\begin{aligned} &\lambda_1 = 1 && \lambda_2 = -1 \\ &v_1 = [\sqrt{3}\ \ 1]^\top && v_2 = [-1\ \ \sqrt{3}]^\top. \end{aligned}$$

Hence the fundamental set is

$$S = \{v_1 e^{t}, v_2 e^{-t}\}$$

(d) It is determined by (a, b). If (a, b) is aligned with v_1 then $(x_1(t), x_2(t))$ will converge to 0, and if (a, b) is aligned with v_2, then it will go to $(-\infty, +\infty)$, moreover, if it is aligned with $-v_2$ then it will converge to $(+\infty, -\infty)$. For other positions, it will move curly and finally converge to either $(-\infty, \infty)$ or $(\infty, -\infty)$ depending on the value of (a, b).
Remark: You should be able to determine the long term behaviour by the direction field!

Problem 6. (20 pts) In this problem we explore the different type of linear systems of differential equations with repeated eigenvalues.

(a) Provide a fundamental set of solutions for the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

(b) Find a fundamental set of solutions for the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 10 & -2 \\ 18 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

(c) Find the unique solution to the above 2×2 system which satisfies

$$(x_1(0), x_2(0)) = (1, 0).$$

Solution:

(a) The eigenvalues and eigenvectors are:

$$\begin{array}{lll} \lambda_1 = 2 & \lambda_2 = 2 & \lambda_3 = -4 \\ v_1 = [1 \;\; 1 \;\; 0]^\top & v_2 = [-2 \;\; 0 \;\; 1]^\top & v_3 = [-1/2 \quad -3/2 \;\; 1]^\top. \end{array}$$

Note that we have a repeated eigenvalue and in this case v_1 and v_2 are both associated to $\lambda = 2$. Therefore the fundamental set is

$$S = \{v_1 e^{2t}, v_2 e^{2t}, v_3 e^{-4t}\}$$

(b) For this matrix we have only one eigenvalue with multiplicity 2 and one eigenvector:

$$\lambda = 4 \quad v = [1 \;\; 3]^\top.$$

Hence the fundamental set is $\{ve^{4t}, (tv + u)e^{4t}\}$, where $u = [1/6 \quad 0]^\top$ is the solution to $(A - \lambda I)u = v$.

(c) Apply the initial condition and we have:

$$c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which implies $c_1 = 0, c_2 = 6$, hence the solution is:

$$x(t) = 6\left(t\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1/6\\0\end{bmatrix}\right)e^{4t}.$$

Problem 7. (20 pts) Consider the following non-homogeneous system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} -1 \\ -6 \\ -5 \end{pmatrix}$$

(a) Find all solutions to the above linear non-homogeneous system.

(b) Find the unique solution such that

$$(x_1(0), x_2(0), x_3(0)) = (0, 0, 0).$$

Solution:

(a) Let's first find a particular solution: since the non-homogeneous part is just a constant vector. We have a good reason to guess that we have a constant, particular solution: let the original equation be rewritten as:

$$x' = Ax - b.$$

When x is a constant we have $x' = 0$, then it turns into a linear system: $Ax = b$. Therefore the particular solution we seek for is:

$$x_{particular} = A^{-1}b = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix}1\\6\\5\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}.$$

And for the homogeneous part, eigenvalues and eigenvectors of A are:

$$\begin{array}{lll} \lambda_1 = -1 & \lambda_2 = -2 & \lambda_3 = 3 \\ v_1 = [-1\ \ 0\ \ 1]^\top & v_2 = [1\ \ -1\ \ 3] & v_3 = [1\ \ 4\ \ 3]^\top. \end{array}$$

Therefore all solutions can be written as:

$$\begin{aligned} x &= x_{homogeneous} + x_{particular} \\ &= c_1\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t} + c_2\begin{bmatrix}1\\-1\\3\end{bmatrix}e^{-2t} + c_3\begin{bmatrix}1\\4\\3\end{bmatrix}e^{3t} + \begin{bmatrix}1\\2\\1\end{bmatrix}. \end{aligned}$$

(b) Apply the initial condition and we have a linear system for c_1, c_2, c_3:

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 4 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-1\\-2\\-1\end{bmatrix},$$

which implies: $c_1 = 1/2, c_2 = 0, c_3 = -1/2$.