MAT 22B: PROBLEM SET 6

DUE TO FRIDAY DEC 6 AT 10:00AM

ABSTRACT. This is the sixth problem set for the Differential Equations Course in the Fall Quarter 2019. It is due Friday Dec 6 at 10:00am via submission to Gradescope.

Purpose: The goal of this assignment is to practice solving linear systems of differential equations via the exponential of a matrix, drawing phase portraits and linearizing systems. In particular, we would like to become familiar with the use of diagonalization in the implementation of the above processes.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following matrices, compute their exponential **by using the definition** of the exponential of a matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}, \begin{pmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \\ -5 & -5 & -5 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 2. Draw the phase portrait of the following 3-dimensional system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Problem 3. (20 pts) **Practicing The Exponential Solution**. Consider the following Initial Value Problem:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) Find the unique solution to the above system by using the exponential $\exp(At)$, where A is the 2×2 matrix defining the linear system.

(b) Find the unique solution to the above linear system but now using the initial condition $x(4) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Hint: There is no need to recompute, use your solution in Part (a) and adjust.

(b) Solve the Initial Value Problem in Part (a) *without* using the exponential matrix and compare your answers.

(c) Let $a, b \in \mathbb{R}$. Does the Initial Value Problem above *always* have a solution for any choice of Initial Value $x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$?

Problem 4. (20 pts) **Drawing Phase Portraits**. Consider the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 9 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 5 \\ -5 & -1 \end{pmatrix}, \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}.$$

(a) For each of the following seven matrices A above draw the phase portrait of the associated linear system $x'(t) = A \cdot x(t)$.

(b) For each of the seven systems above, describe the long-term behavior of the unique solution with starts at $x(0) = (1, 1)^t$.

Problem 5. (20 pts) **Variation of Parameters.** Consider the following non-homogeneous linear system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} e^{2t} \\ -e^t \end{pmatrix}$$

(a) Find a particular solution to the above linear system.

(b) Give the general solution of the linear system.

(c) Does there exist a non-zero constant solution ?

Problem 6. (20 pts) **Variation of Parameters II.**
Solve the following non-homogeneous Initial Value Problem:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} - \begin{pmatrix} 15te^{-2t} \\ 5te^{-2t} \end{pmatrix}, \quad x(0) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}.$$

Problem 7. (20 pts) **Linearization of Systems.** Consider the following 2-dimensional non-linear system:

$$x_1'(t) = 3x_1(t) - x_2(t)^2,$$
$$x_2'(t) = -x_1(t) + \sin(x_2(t)).$$

(a) Find all constant solutions to the above system of differential equations. (There should be two of them.)

(b) Linearize the system at each of the constant solutions.

(c) Draw qualitatively a phase portrait for the system.