University of California Davis **Name (Print):** ____________________

Differential Equations MAT 22B **Student ID (Print):** ____________________

Final Examination **December 9 2019**

Time Limit: 120 Minutes

This examination document contains 10 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following differential equation:

$$y''(t) + 2y'(t) + 10y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 1.$$

(a) (5 points) Find the unique solution to the above Initial Value Problem for the external force $g(t) = 0$ and determine whether the system is overdamped, critically damped or underdamped.

(b) (5 points) Find the unique solution to the above Initial Value Problem for the external force $g(t) = 1 + 25t^2$.

(c) (5 points) Find the unique solution to the above Initial Value Problem for the external force $g(t) = \delta(t - 10)$ and qualitatively plot this solution.

(d) (5 points) Plot qualitatively the unique solution to the above Initial Value Problem for $g(t) = \delta(t - 10) + \delta(t - 50) + \delta(t - 100) + \delta(t - 150)$.

2. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 10 & -2 & 1 \\ 18 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

(a) (5 points) Find a fundamental set of solutions to the differential system.

(b) (5 points) Find *all* solutions to the system of differential equations above.

(c) (5 points) Find the solutions to the system which satisfy $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(d) (5 points) Compute the long-term behavior of *all* non-zero solutions $\vec{x}(t)$.

3. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

(a) (8 points) Find a particular solution $\vec{x}_p(t)$ to the linear system above.

(b) (4 points) Find *all* solutions to the linear system above.

(c) (4 points) Find *all* solutions with $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(d) (4 points) Are there any constant solutions to the linear system ?
(Justify your answer: if yes, give at least one, if no, argue why that is the case.)

4. (20 points) Let $\alpha \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (4 points) Find the interval of values for $\alpha \in \mathbb{R}$ such that the phase-portrait for the linear system above consists of a spiraling behavior[1].

(b) (4 points) For which values of $\alpha \in \mathbb{R}$ does *every* solution to the above linear system have long-term behavior equal to zero ?

[1]Concentric circles are also considered spirals.

(c) (4 points) Describe the long-term behavior of the unique solution $\vec{x}(t)$ to the linear system above such that $\vec{x}(0) = \begin{pmatrix} -15 \\ 3 \end{pmatrix}$ for the value $\alpha = -2$.

(d) (4 points) Plot qualitatively the phase-portrait of the system for $\alpha = 5$.

(e) (4 points) Plot qualitatively the phase-portrait of the system for $\alpha = 0$.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**.

(a) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(1) True. (2) False.

(b) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}$ is the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(1) True. (2) False.

(c) (2 points) There exist an autonomous first-order differential equation with infinitely many stable solutions.

(1) True. (2) False.

(d) (2 points) If an autonomous first-order differential equation has three stable solutions then it must have an unstable solution.

(1) True. (2) False.

(e) (2 points) The local error in Euler's method with step $h = 0.01$ is of order 10^{-4}:

(1) True. (2) False.

(f) (2 points) A linear system of differential equations $\vec{x}(t)' = A\vec{x}(t)$ with $\det(A) \neq 0$ does not have a non-zero constant solution.

(1) True. (2) False.

(g) (2 points) A linear system $\vec{x}(t)' = A\vec{x}(t) + g(t)$ with $\det(A) \neq 0$ cannot have a non-zero constant solution even if $g(t)$ is constant.

(1) True. (2) False.

(h) (2 points) The Laplace transform $\mathcal{L}(f)(s)$ of $f(t) = e^{t^2}$ is $\mathcal{L}(e^{t^2})(s) = s^{-2}$.

(1) True. (2) False.

(i) (2 points) The vector $e^{At} \cdot x_0$ solves the Initial Value Problem $\vec{x}(t)' = A\vec{x}(t)$ with initial condition $x(0) = x_0$ if and only if A is diagonalizable.

(1) True. (2) False.

(j) (2 points) The non-linear system of two differential equations

$$x'(t) = y(t) - x^3(t) + x(t), \quad y'(t) = x(t)^2(\cos(y(t)) + 2))$$

has at least one constant solution.

(1) True. (2) False.