University of California Davis **Name (Print):** ______________________

Differential Equations MAT 22B **Student ID (Print):** ______________________

Final Examination **December 9 2019**

Time Limit: 120 Minutes

This examination document contains 10 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) Consider the following differential equation:

$$y''(t) + 2y'(t) + 10y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 1.$$

(a) (5 points) Find the unique solution to the above Initial Value Problem for the external force $g(t) = 0$ and determine whether the system is overdamped, critically damped or underdamped.

Solution: The characteristic equation has roots $\lambda_{\pm} = -1 \pm 3i$. Thus the general solution reads:

$$y(t) = e^{-t}(C_1 \sin(3t) + C_2 \cos(3t)).$$

The condition $y(0) = 0$ implies $C_2 = 0$, and $y'(0) = 1$ implies $3C_1 = 1$. Hence the unique solution is

$$y(t) = \frac{1}{3} e^{-t} \sin(3t).$$

(b) (5 points) Find the unique solution to the above Initial Value Problem for the external force $g(t) = 1 + 25t^2$.

Solution: Let us apply the Method of Undetermined Coefficients. Our ansatz is $y_p(t) = A + Bt + Ct^2$. Plugging into the Differential Equation we obtain that

$$A = -1/5, \quad B = -1, \quad C = 5/2.$$

Thus the general solution reads:

$$y(t) = e^{-t}(C_1 \sin(3t) + C_2 \cos(3t)) - \frac{1}{5} - t + \frac{5t^2}{2}.$$

The condition $y'(0) = 1$ implies $C_1 = 11/15$, and $y(0) = 0$ implies $C_2 = 1/5$. Hence the unique solution is

$$y(t) = e^{-t}\left(\frac{11}{15} \sin(3t) + \frac{1}{5} \cos(3t)\right) - \frac{1}{5} - t + \frac{5t^2}{2}.$$

(c) (5 points) Find the unique solution to the above Initial Value Problem for the external force $g(t) = \delta(t - 10)$ and qualitatively plot this solution.

Solution: Let us use the Laplace transform. The Laplace transform of the differential equation reads:

$$\mathcal{L}(y)(s^2 + 2s + 10) = e^{-10s}.$$

Thus we obtain that

$$\mathcal{L}(y) = \frac{e^{-10s}}{s^2 + 2s + 10} = \frac{e^{-10s}}{3} \cdot \frac{3}{(s+1)^2 + 9}.$$

By taking the anti-Laplace transform we obtain

$$y(t) = \frac{1}{3}e^{-t}\sin(3t) + \frac{1}{3}e^{-t+10}u_0(3t - 30)\sin(3t - 30).$$

(d) (5 points) Plot qualitatively the unique solution to the above Initial Value Problem for $g(t) = \delta(t - 10) + \delta(t - 50) + \delta(t - 100) + \delta(t - 150)$.

Solution: The solution is depicted in Figure 1.

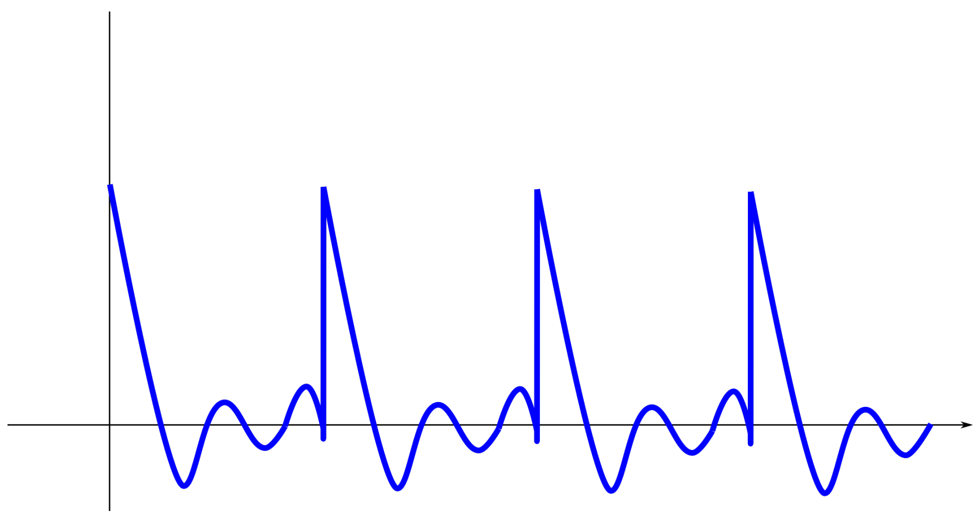

Figure 1: The qualitative plot for the solution of Part 1.(d).

2. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 10 & -2 & 1 \\ 18 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

(a) (5 points) Find a fundamental set of solutions to the differential system.

Solution: The eigenvalues are $\lambda_1 = \lambda_2 = 4$ with multiplicity two and $\lambda_3 = 1$ of multiplicity one. The case of $\lambda_1 = 4$ is defective. An eigenvector for $\lambda_1 = 4$ is $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, and a generalized eigenvector is $\begin{pmatrix} 1/6 \\ 0 \\ 0 \end{pmatrix}$. An eigenvector for $\lambda_3 = 1$ is $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$. Thus a fundamental set of solutions is given by

$$e^{4t}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad e^{4t}(\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \cdot t + \begin{pmatrix} 1/6 \\ 0 \\ 0 \end{pmatrix}), \quad e^{t} \cdot \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}.$$

(b) (5 points) Find *all* solutions to the system of differential equations above.

Solution: The general solution of the system is obtained by multiplying a fundamental matrix by a constant vector:

$$\vec{x}(t) = \begin{pmatrix} e^{4t} & e^{4t}(t + 1/6) & e^{t} \\ 3e^{4t} & 3e^{4t}t & 6e^{t} \\ 0 & 0 & 3e^{t} \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

(c) (5 points) Find the solutions to the system which satisfy $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Solution: The general solution at $t = 0$ reads:

$$\vec{x}(t) = \begin{pmatrix} 1 & 1/6 & 1 \\ 3 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

Thus we obtain $C_1 = -2/3, C_2 = 2$ and $C_3 = 1/3$.

(d) (5 points) Compute the long-term behavior of *all* non-zero solutions $\vec{x}(t)$.

Solution: The eigenvalues are all real and positive, thus the long-term behaviour of any non-zero solution is infinity, i.e. it does not exist.

3. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

(a) (8 points) Find a particular solution $\vec{x}_p(t)$ to the linear system above.

Solution: Let us apply variation of parameters. A particular solution will be given by $\vec{x}_p(t) = \Psi(t)\int \Psi(t)^{-1}g(t)$, where $g(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$. The eigenvalues of the system are $\lambda_1 = 1$ and $\lambda_2 = -1$, with eigenvectors $\xi_1 = (1,1)^t$ and $\xi_{-1} = (1,3)^t$ respectively. A fundamental matrix for the system is thus

$$\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}.$$

If one looks for $\Psi(t)$ such that $\Psi(0) = \mathrm{Id}$ this reads A fundamental matrix for the system is thus

$$\begin{pmatrix} \frac{1}{2}(3e^t - e^{-t}) & -\frac{1}{2}(e^t - e^{-t}) \\ \frac{3}{2}(e^t - e^{-t}) & -\frac{1}{2}(e^t - 3e^{-t}) \end{pmatrix}.$$

The particular solution is thus obtained by integrating the product

$$\begin{pmatrix} \frac{1}{2}(3e^t - e^{-t}) & -\frac{1}{2}(e^t - e^{-t}) \\ \frac{3}{2}(e^t - e^{-t}) & -\frac{1}{2}(e^t - 3e^{-t}) \end{pmatrix} \cdot \begin{pmatrix} e^t \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(3e^{2t} - 1) - \frac{t}{2}(e^t - e^{-t}) \\ \frac{3}{2}(e^{2t} - 1) - \frac{t}{2}(e^t - 3e^{-t}) \end{pmatrix},$$

and multiplying by $\Psi(t)$.

(b) (4 points) Find *all* solutions to the linear system above.

Solution: Consider the particular solution $\vec{x}_p(t)$ in Part (a). The general solution of the non-homogeneous linear system is:

$$\vec{x}(t) = \begin{pmatrix} \frac{1}{2}(3e^t - e^{-t}) & -\frac{1}{2}(e^t - e^{-t}) \\ \frac{3}{2}(e^t - e^{-t}) & -\frac{1}{2}(e^t - 3e^{-t}) \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \vec{x}_p(t).$$

(c) (4 points) Find *all* solutions with $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Solution: Consider the solution in Part (b). The general solution of the non-homogeneous linear system at $t = 0$ reads:

$$\vec{x}(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus $C_1 = -1$ and $C_2 = 0$.

(d) (4 points) Are there any constant solutions to the linear system ?
(Justify your answer: if yes, give at least one, if no, argue why that is the case.)

Solution: Consider the general solution in Part (b). If a constant solution exist, we must have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

Since the matrix of the system is invertible, any constant solution would be exactly given by the inverse of the matrix times $g(t)$, which is itself not constant. Hence, no constant solution exists.

4. (20 points) Let $\alpha \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (4 points) Find the interval of values for $\alpha \in \mathbb{R}$ such that the phase-portrait for the linear system above consists of a spiraling behavior[1].

Solution: The condition is equivalent to the eigenvalues being imaginary. The eigenvalues are the roots of $\lambda^2 - \alpha\lambda + 1$. The discriminant is negative if and only if $|\alpha| < 2$. Thus the required interval of values is $\alpha \in (-2, 2)$.

(b) (4 points) For which values of $\alpha \in \mathbb{R}$ does *every* solution to the above linear system have long-term behavior equal to zero ?

Solution: The condition is equivalent to the real part of both eigenvalues is negative. This is equivalent to $\alpha < 0$.

[1] Concentric circles are also considered spirals.

(c) (4 points) Describe the long-term behavior of the unique solution $\vec{x}(t)$ to the linear system above such that $\vec{x}(0) = \begin{pmatrix} -15 \\ 3 \end{pmatrix}$ for the value $\alpha = -2$.

Solution: For $\alpha = -2$ both eigenvalues are equal to -1. Thus every solution converges to 0 in the long-term behavior.

(d) (4 points) Plot qualitatively the phase-portrait of the system for $\alpha = 5$.

Solution: In this case of $\alpha = 5$ both eigenvalues are positive. Thus, the phase-portrait consists of straight lines from the origin, all pointing outwards.

(e) (4 points) Plot qualitatively the phase-portrait of the system for $\alpha = 0$.

Solution: In this case of $\alpha = 0$ both eigenvalues are purely imaginary and the phase-portrait consists of concentric circles.

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**.

(a) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(1) **True**. (2) False.

(b) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}$ is the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(1) **True**. (2) False.

(c) (2 points) There exist an autonomous first-order differential equation with infinitely many stable solutions.

(1) **True**. (2) False.

(d) (2 points) If an autonomous first-order differential equation has three stable solutions then it must have an unstable solution.

(1) **True**. (2) False.

(e) (2 points) The local error in Euler's method with step $h = 0.01$ is of order 10^{-4}:

(1) **True**. (2) False.

(f) (2 points) A linear system of differential equations $\vec{x}(t)' = A\vec{x}(t)$ with $\det(A) \neq 0$ does not have a non-zero constant solution.

(1) **True**. (2) False.

(g) (2 points) A linear system $\vec{x}(t)' = A\vec{x}(t) + g(t)$ with $\det(A) \neq 0$ cannot have a non-zero constant solution even if $g(t)$ is constant.

(1) True. (2) **False**.

(h) (2 points) The Laplace transform $\mathcal{L}(f)(s)$ of $f(t) = e^{t^2}$ is $\mathcal{L}(e^{t^2})(s) = s^{-2}$.

(1) True. (2) **False**.

(i) (2 points) The vector $e^{At} \cdot x_0$ solves the Initial Value Problem $\vec{x}(t)' = A\vec{x}(t)$ with initial condition $x(0) = x_0$ if and only if A is diagonalizable.

(1) True. (2) **False**.

(j) (2 points) The non-linear system of two differential equations

$$x'(t) = y(t) - x^3(t) + x(t), \quad y'(t) = x(t)^2(\cos(y(t)) + 2))$$

has at least one constant solution.

(1) **True**. (2) False.