University of California Davis **Name (Print):** ____________________

Differential Equations MAT 22B **Student ID (Print):** ____________________

Midterm Examination - Solutions **October 25 2019**

Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following first-order differential equation:

$$y'(t) - 5y(t) = e^{5t}.$$

(a) (10 points) Find *all* solutions to the differential equation.

Solution: This first-order linear non-homogeneous differential equation can be solved by the method of integrating factors. The integrating factor is $\mu(t) = e^{-5t}$. Multiplying the ODE by $\mu(t)$ on both sides we obtain:

$$(e^{-5t}y(t))' = 1.$$

Thus $(e^{-5t}y(t)) = t + C$, $C \in \mathbb{R}$, and the general solution is:

$$y(t) = e^{5t}(t + C).$$

(b) (5 points) Find the solution $y_0(t)$ to the differential equation satisfying $y_0(0) = 4$.

Solution: This correspond to the condition

$$4 = y(0) = e^{5\cdot 0}(0 + C) \iff C = 4.$$

Hence, the unique solution is $y_0(t) = e^{5t}(t + 4)$.

(c) (5 points) Compute the long-term behavior of *all* solutions $y(t)$.

Solution: The long-term behaviour is the limit

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} e^{5t}(t + C) = \infty.$$

Thus the long-term behavior is infinity, i.e. it does not exist.

2. (20 points) Consider the following first-order differential equation:

$$y'(t) = y(y-1)(y+1)(y-3)(y-5)^2.$$

(a) (7 points) Find the constant solutions of the above differential equation and classify them into *stable*, *unstable* and *semistable*.

Solution: Since this is an autonomous differential equation, the constant solutions, which satisfy $y'(t) = 0$, correspond to the zeroes of the function $f(y) = y(y-1)(y+1)(y-3)(y-5)^2$. The constant solutions are thus $\{-1, 0, 1, 3, 5\}$.

The constant solutions $y(t) \equiv -1$, $y(t) \equiv 1$ are stable, $y(t) \equiv 0$ and $y(t) \equiv 3$ are unstable and $y(t) \equiv 5$ is semistable.

(b) (7 points) Let $y_1(t), y_2(t), y_3(t)$ be the solutions of the above differential equation which satisfy $y_1(0) = 2$, $y_2(6) = 3$ and $y_3(0) = 4$, respectively. Compute their long-term behaviors

$$\lim_{t\to\infty} y_1(t), \qquad \lim_{t\to\infty} y_2(t), \qquad \lim_{t\to\infty} y_3(t).$$

Solution: Their long-term behaviors correspond to constant stable solutions. The exact constant depends on the initial value. The solution $y_1(t)$ will converge to 1, the solution $y_2(t)$ is itself constant to 3 and thus converges to 3 and the long-term behavior of $y_3(t)$ will be 5.

(c) (6 points) Use Euler's method with step $h = 0.1$ to approximate the value of $y_3(0.2)$, where $y_3(t)$ is the solution to the differential equation above satisfying $y_3(0) = 4$.

Solution: The first iteration yields

$$y_3(0.1) \approx 4 + 0.1 \cdot f(4, 0.1) = 4 + 0.1 \cdot f(4) = 4 + 0.1 \cdot 60 = 10.$$

The second iteration gives

$$y_3(0.2) \approx 10 + 0.1 \cdot f(10) = 10 + 0.1 \cdot 173250 = 10 + 17325 = 173260.$$

(Solution will be fully graded if 173250 left as $7 \cdot 9 \cdot 10 \cdot 11 \cdot 5^2$.)

3. (20 points) Consider the second-order differential equation:

$$y''(t) + 9y'(t) + 20y(t) = t + 2.$$

(a) (10 points) Find *one* solution to the differential equation.

Solution: Let us apply the method of undetermined coefficients. The guess is $y(t) = At + B$. By plugging

$$y'(t) = A, \quad y''(t) = 0,$$

into the differential equation, the two conditions for A, B are

$$20A = 1, \quad 9A + 20B = 2.$$

Thus one solution is

$$y_p(t) = \frac{t}{20} + \frac{31}{400}.$$

(b) (5 points) Find *all* solutions to the differential equation above.

Solution: Since the equation is linear, the principle of superposition implies that *all* will be of the form:

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t), \quad C_1, C_2 \in \mathbb{R},$$

where $y_1(t), y_2(t)$ are two linearly independent solutions of the homogeneous problem. In this case, the homogeneous problem can be solved by its characteristic equation, whose roots are $\lambda_1 = -4$ and $\lambda_2 = -5$. This yields $y_1(t) = e^{-4t}$ and $y_2(t) = e^{-5t}$, which are linearly independent since $W(e^{-4t}, e^{-5t}) = -e^{-9t} \neq 0$. Thus the general solution of the differential equation will be:

$$y(t) = C_1 e^{-4t} + C_2 y_2(t) + \frac{t}{20} + \frac{31}{400}, \quad C_1, C_2 \in \mathbb{R}.$$

(c) (5 points) Consider the damped harmonic oscillator given by

$$y''(t) + 9y'(t) + 20y(t) = 0.$$

Determine whether the system is *underdamped*, *critically damped* or *overdamped*.

Solution: The roots are real and distinct, the system is thus **overdamped**.

4. (20 points) Consider the second-order differential equation:

$$2t^2y''(t) + 3ty'(t) - 15y(t) = 0, \quad t > 0.$$

(a) (5 points) Show that $y_1(t) = t^{-3}$ is a solution to the above differential equation.

Solution: This is plug and check, as follows. Start by computing $y_1'(t) = -3t^{-4}$ and $y_1''(t) = 12t^{-4}$. Then we have

$$t^2y''(t) + 3ty'(t) - 15y(t) = t^2 \cdot (12t^{-4}) + (3t) \cdot -3t^{-4} - 15t^{-3} = 0 \cdot t^{-3} = 0,$$

as was required for us to verify.

(b) (10 points) Find a second solution $y_2(t)$ which is linearly independent with $y_1(t)$.

Solution: This can be solved by Euler type ansatz or via reduction of coefficients. The former does not require knowing $y_1(t)$ in Part (a), the latter does. Let us use Euler type ansatz $y(t) = t^r$, whose characteristic equation is:

$$r(r-1) + \frac{3r}{2} - \frac{15}{2}.$$

The two roots are $r_1 = -3$, corresponding to $y_1(t)$ in Part (a) and $r_2 =$. Thus, a second linearly independent solution is $y_2(t) = t^{5/2}$. Their Wronskian $W(t^{-3}, t^{5/2})$ is of the form t^s, for some $s \in \mathbb{R}$, and thus non-zero since t is positive.

(c) (5 points) Find *all* solutions to the following differential equation:

$$2t^2y''(t) + 3ty'(t) - 15y(t) = 15, \quad t > 0.$$

Solution: Since this is a linear non-homogeneous equation, the general solution will be of the form:

$$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t), \quad C_1, C_2 \in \mathbb{R},$$

where $y_1(t), y_2(t)$ are two linearly independent solutions of the homogeneous problem, and $y_p(t)$ a particular solution to the non-homogeneous problem. In this case $y_1(t)$ is the solution in Part (a) and $y_2(t)$ is the solution found in Part (b). It thus suffices to find a particular solution $y_p(t)$.

The constant function $y_p(t) = -1$ solves the problem, thus the general solution to the problem is:

$$y(t) = C_1t^{-3} + C_2t^{5/2} - 1, \quad C_1, C_2 \in \mathbb{R}.$$

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**.

(a) (2 points) An autonomous first-order differential equation must have as many stable constant solutions as unstable constant solutions.

(1) True. (2) **False**.

(b) (2 points) If $y_1(t)$ and $y_2(t)$ solve a first-order differential equation, then their sum $y_1(t) + y_2(t)$ also solves the same differential equation.

(1) True. (2) **False**.

(c) (2 points) Solutions to underdamped homogeneous systems must have infinitely many zeroes.

(1) **True**. (2) False.

(d) (2 points) If $y_1(t) = \sin(3t)$ solves a linear second-order differential equation, then $y_2(t) = \cos(3t)$ also solves the same differential equation.

(1) True. (2) **False**.

(e) (2 points) The local error in Euler's method with step 0.01 is of the order of 0.0001.

(1) **True**. (2) False.

(f) (2 points) An autonomous first-order differential equation cannot have infinitely many semistable constant solutions.

(1) True. (2) **False**.

(g) (2 points) The initial value problem given by a second-order differential equation on $y(t)$ and two initial conditions $y(t_0) = y_0$ and $y(t_1) = y_1$ always has a solution.

(1) True. (2) **False**.

(h) (2 points) The Wronskian of $y_1(t) = e^{-t}$ and $y_2(t) = e^{t}$ is constant equal to 2.

(1) **True**. (2) False.

(i) (2 points) The global error in Euler's method is smaller than the local error.

(1) True. (2) **False**.

(j) (2 points) The Wronskian of $y_1(t) = 1 - t$ and $y_2(t) = t^3$ is $W(y_1, y_2) = t^2$.

(1) True. (2) **False**.