This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may **not** use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

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1. (20 points) Consider the following linear system of differential equations:

\[
\begin{pmatrix}
x'_1(t) \\
x'_2(t) \\
x'_3(t)
\end{pmatrix} =
\begin{pmatrix}
-1 & 1 & 5 \\
0 & -1 & 4 \\
0 & 0 & -5
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix}.
\]

(a) (5 points) Find a fundamental set of solutions to the differential system.

(b) (5 points) Find all solutions to the system of differential equations above.

(c) (5 points) Find the solutions to the system which satisfy

\[
\begin{pmatrix}
x_1(0) \\
x_2(0) \\
x_3(0)
\end{pmatrix} =
\begin{pmatrix}
12 \\
0 \\
1
\end{pmatrix}.
\]

(d) (5 points) Compute the long-term behavior of all solutions \( \vec{x}(t) \).
2. (20 points) Let $\gamma \in \mathbb{R}$ and consider the following system of differential equations:

\[
\begin{pmatrix}
    x'_1(t) \\
    x'_2(t)
\end{pmatrix} = \begin{pmatrix}
    0 & 1 \\
    -1 & \gamma
\end{pmatrix} \begin{pmatrix}
    x_1(t) \\
    x_2(t)
\end{pmatrix}.
\]

(a) (5 points) Draw the phase-portrait of the system for the three values $\gamma \in \{-1, 0, 1\}$.

(b) (5 points) Set $\gamma = 3$, compute the long-term behaviour of the unique solution $\vec{x}(t)$ with the initial condition $x(0) = \begin{pmatrix}
    0.002 \\
    -0.03
\end{pmatrix}$.

(c) (5 points) Is there a value of $\gamma \in \mathbb{R}$ such that the phase-portrait is not a spiral but the long-term behaviour of all solutions is still zero?

(d) (5 points) Find all the values $\gamma \in \mathbb{R}$ for which phase-portrait of the system behaves as a spiral. (Concentric circles are considered spirals as well.)
3. (20 points) Consider the following differential equation:

\[ y''(t) + y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

(a) (10 points) Find the unique solution to the above differential equation for the external force \( g(t) = \delta(t - 2) + \delta(t - 15) \) and plot it.

(b) (5 points) Find the unique solution to the above differential equation for the external force \( g(t) = \delta(t - \pi) + \delta(t - 2\pi) \). How does its long-term behavior differ from the solution in Part (a)?

(c) (5 points) Find an external force \( g(t) \) such that the above Initial Value Problem has a solution which qualitatively looks as in Figure 1.

![Figure 1: The plot of the solution for Part (c).](image)
4. (20 points) Consider the non-homogeneous linear system of differential equations:

\[
\begin{pmatrix}
    x_1'(t) \\
    x_2'(t)
\end{pmatrix} = \begin{pmatrix}
    1 & 2 \\
    2 & 1
\end{pmatrix} \begin{pmatrix}
    x_1(t) \\
    x_2(t)
\end{pmatrix} + \begin{pmatrix}
    2e^{4t} \\
    e^{4t}
\end{pmatrix}.
\]

(a) (10 points) Find a particular solution \( \vec{x}_p(t) \) to the linear system above.

(b) (5 points) Find all solutions to the linear system above.

(c) (5 points) Find all solutions with \( x_1(0) = 8/5 \).
5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (2 points) The exponential of the matrix \( \begin{pmatrix} 2t & 0 \\ 0 & -4t \end{pmatrix} \) is \( \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-4t} \end{pmatrix} \).

(1) True. (2) False.

(b) (2 points) The exponential of the matrix \( \begin{pmatrix} t & t \\ 0 & t \end{pmatrix} \) is \( \begin{pmatrix} e^t & e^t \\ 0 & e^t \end{pmatrix} \).

(1) True. (2) False.

(c) (2 points) There exist a autonomous first-order differential equation with infinitely many semistable solutions.

(1) True. (2) False.

(d) (2 points) If an autonomous first-order differential equation has two stable solutions then it must have an unstable solution.

(1) True. (2) False.

(e) (2 points) The local error in Euler’s method with step \( h = 0.01 \) is of order \( 10^{-1} \):

(1) True. (2) False.

(f) (2 points) A homogeneous linear system of differential equations \( \vec{x}(t)' = A\vec{x}(t) \) never has a non-zero constant solution.

(1) True. (2) False.

(g) (2 points) A linear system of differential equations \( \vec{x}(t)' = A\cdot\vec{x}(t) + g(t) \) might have non-zero constant solutions for certain \( g(t) \).

(1) True. (2) False.

(h) (2 points) The Laplace transform \( \mathcal{L}(f)(s) \) of a differentiable function \( f : \mathbb{R} \rightarrow \mathbb{R} \) always exist and is differentiable with respect to \( s \).

(1) True. (2) False.

(i) (2 points) The Laplace transform \( \mathcal{L} \) is a linear transformation.

(1) True. (2) False.

(j) (2 points) The non-linear system of two differential equations

\[
x'(t) = y(t) - x^3(t) + x(t), \quad y'(t) = -x(t)e^{y(t)}
\]

has exactly two constant solutions.

(1) True. (2) False.