University of California Davis **Name (Print):** ____________

Differential Equations MAT 22B **Student ID (Print):** ____________

Sample Final Examination **December 9 2019**

Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 5 \\ 0 & -1 & 4 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

(a) (5 points) Find a fundamental set of solutions to the differential system.

(b) (5 points) Find *all* solutions to the system of differential equations above.

(c) (5 points) Find the solutions to the system which satisfy $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix}$.

(d) (5 points) Compute the long-term behavior of *all* solutions $\vec{x}(t)$.

2. (20 points) Let $\gamma \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \gamma \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (5 points) Draw the phase-portrait of the system for the three values $\gamma \in \{-1, 0, 1\}$.

(b) (5 points) Set $\gamma = 3$, compute the long-term behaviour of the unique solution $\vec{x}(t)$ with the initial condition $x(0) = \begin{pmatrix} 0.002 \\ -0.03 \end{pmatrix}$.

(c) (5 points) Is there a value of $\gamma \in \mathbb{R}$ such that the phase-portrait is *not* a spiral but the long-term behaviour of *all* solutions is still zero ?

(d) (5 points) Find all the values $\gamma \in \mathbb{R}$ for which phase-portrait of the system behaves as a spiral. (Concentric circles are considered spirals as well.)

3. (20 points) Consider the following differential equation:

$$y''(t) + y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

(a) (10 points) Find the unique solution to the above differential equation for the external force $g(t) = \delta(t-2) + \delta(t-15)$ and plot it.

(b) (5 points) Find the unique solution to the above differential equation for the external force $g(t) = \delta(t-\pi) + \delta(t-2\pi)$. How does its long-term behavior differ from the solution in Part (a) ?

(c) (5 points) Find an external force $g(t)$ such that the above Initial Value Problem has a solution which qualitatively looks as in Figure 1.

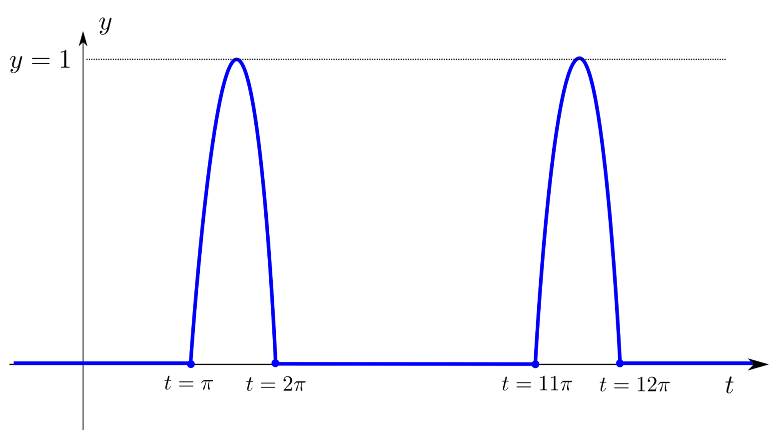

Figure 1: The plot of the solution for Part (c).

4. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}.$$

(a) (10 points) Find a particular solution $\vec{x}_p(t)$ to the linear system above.

(b) (5 points) Find *all* solutions to the linear system above.

(c) (5 points) Find *all* solutions with $x_1(0) = 8/5$.

5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (2 points) The exponential of the matrix $\begin{pmatrix} 2t & 0 \\ 0 & -4t \end{pmatrix}$ is $\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-4t} \end{pmatrix}$.

(1) True. (2) False.

(b) (2 points) The exponential of the matrix $\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$ is $\begin{pmatrix} e^{t} & e^{t} \\ 0 & e^{t} \end{pmatrix}$.

(1) True. (2) False.

(c) (2 points) There exist a autonomous first-order differential equation with infinitely many semistable solutions.

(1) True. (2) False.

(d) (2 points) If an autonomous first-order differential equation has two stable solutions then it must have a unstable solution.

(1) True. (2) False.

(e) (2 points) The local error in Euler's method with step $h = 0.01$ is of order 10^{-1}:

(1) True. (2) False.

(f) (2 points) A homogeneous linear system of differential equations $\vec{x}(t)' = A\vec{x}(t)$ never has a non-zero constant solution.

(1) True. (2) False.

(g) (2 points) A linear system of differential equations $\vec{x}(t)' = A \cdot \vec{x}(t) + g(t)$ might have non-zero constant solutions for certain $g(t)$.

(1) True. (2) False.

(h) (2 points) The Laplace transform $\mathcal{L}(f)(s)$ of a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ always exist and is differentiable with respect to s.

(1) True. (2) False.

(i) (2 points) The Laplace transform $\mathcal{L}$ is a linear transformation.

(1) True. (2) False.

(j) (2 points) The non-linear system of two differential equations

$$x'(t) = y(t) - x^3(t) + x(t), \quad y'(t) = -x(t)e^{y(t)}$$

has exactly two constant solutions.

(1) True. (2) False.