University of California Davis **Name (Print):** ____________________
Differential Equations MAT 22B **Student ID (Print):** ____________________

Sample Final Examination II **December 9 2019**
Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} t \\ e^t \\ e^t \end{pmatrix}.$$

(a) (5 points) Show that the columns of the following matrix $\Psi(t)$ are fundamental solutions to the homogeneous system associated to the system above:

$$\Psi(t) = \begin{pmatrix} e^t & e^{2t} & 1 \\ e^t & 0 & -1 \\ e^t & e^{2t} & 0 \end{pmatrix}.$$

(b) (10 points) Find a particular solution to the non-homogeneous system above.

You are welcome to use that $\Psi^{-1}(t) = \begin{pmatrix} e^{-t} & e^{-t} & -e^{-t} \\ -e^{-2t} & -e^{-2t} & 2e^{-2t} \\ 1 & 0 & -1 \end{pmatrix}$,

(c) (5 points) Show that there are no constant solutions to the non-homogeneous system above such that $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

2. (20 points) Let A be the matrix $A = \begin{pmatrix} 16 & -35 \\ 6 & -13 \end{pmatrix}$.

   (a) (5 points) Compute the matrix exponential e^{At}.

   (b) (10 points) Find the unique solution $\vec{x}(t)$ of the initial value problem

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 16 & -35 \\ 6 & -13 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   (c) (5 points) Draw qualitatively the phase-portrait of the system in Part (b).

3. (20 points) Consider the following Initial Value Problem (IVP):

$$y''(t) + y(t) = (1 - t) - u_1(t)(1 - t), \quad y(0) = 1, \quad y'(0) = 0,$$

where $u_1(t)$ is the Heaviside step function centered at $t = 1$.

(a) (5 points) Find the Laplace transform of the above differential equation and isolate $\mathcal{L}(y(t))(s)$ as a function of s.

(b) (10 points) Find the unique solution $y(t)$ to the above Initial Value Problem.

(c) (5 points) Plot qualitatively the solution $y(t)$ to the IVP above for $t \in (-\infty, 1)$. (*Hint: The system reads* $y''(t) + y(t) = (1 - t)$ *for* $t \leq 1$)

4. (20 points) Consider the linear differential equation:

$$t^2 y''(t) + 3ty'(t) + 4y(t) = 0, \quad t > 0.$$

(a) (5 points) Show that the two functions

$$y_1(t) = 12t^{-1}\cos(\sqrt{3}\ln(t)), \quad y_2(t) = 9t^{-1}\sin(\sqrt{3}\ln(t))$$

are solutions to the differential equation above.

(b) (5 points) Compute the Wronskian of the solutions $y_1(t)$ and $y_2(t)$.

(c) (5 points) Find the *all* solutions to the differential equation above.

(d) (5 points) Find the *unique* solution $y(t)$ with $y\left(e^{\frac{\pi}{2\sqrt{3}}}\right) = 1$.

5. (20 points) For each of the sentences below, circle the unique correct answer. By definition, the option “Neither” is the correct answer if the other options are not correct.

   (a) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$ is:

   (1) $\begin{pmatrix} 0 & e^t \\ 0 & 0 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (3) $\begin{pmatrix} e^t & t \\ 0 & e^t \end{pmatrix}$ (4) $\begin{pmatrix} 1 & e^t \\ 0 & 1 \end{pmatrix}$

   (b) (2 points) The phase-portrait of the system

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 22313 \\ -132193 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

   can be qualitatively depicted as:

   (1) Concentric circles (2) Inwards Spiral (3) Outwards Spiral (4) Neither

   (c) (2 points) An autonomous differential equation will always have a:

   (1) Stable solution (2) Unstable solution (3) Semistable solution (4) Neither

   (d) (2 points) The global error in Euler’s method with $h = 0.001$ is of order:

   (1) 10^{-1} (2) 10^{-2} (3) 10^{-3} (4) 10^{-4}

   (e) (2 points) The following function $y(t)$ solves the ODE $y'(t) + y^2(t) = 2t^{-2}$:

   (1) $t + 1$ (2) $(2t^3 - 1)t^{-1}(1 + t^3)^{-1}$ (3) $\exp(t) + t^2$ (4) Neither.