1. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} t \\ e^t \\ e^t \end{pmatrix}.$$

(a) (5 points) Show that the columns of the following matrix $\Psi(t)$ are fundamental solutions to the homogeneous system associated to the system above:

$$\Psi(t) = \begin{pmatrix} e^t & e^{2t} & 1 \\ e^t & 0 & -1 \\ e^t & e^{2t} & 0 \end{pmatrix}.$$

Solution. Check that

$$\Psi'(t) = \begin{pmatrix} e^t & 2e^{2t} & 0 \\ e^t & 0 & 0 \\ e^t & 2e^{2t} & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} e^t & e^{2t} & 1 \\ e^t & 0 & -1 \\ e^t & e^{2t} & 0 \end{pmatrix}$$

(b) (10 points) Find a particular solution to the non-homogeneous system above.

You are welcome to use that $\Psi^{-1}(t) = \begin{pmatrix} e^{-t} & e^{-t} & -e^{-t} \\ -e^{-2t} & -e^{-2t} & 2e^{-2t} \\ 1 & 0 & -1 \end{pmatrix}$,

A particular solution could be calculated using variation of parameters.

$$\Psi(t) \int \Psi(t)^{-1} \begin{pmatrix} t \\ e^t \\ e^t \end{pmatrix} = \begin{pmatrix} -2e^t + 1/2 * t^2 - 1/2 * t - 3/4 \\ e^t - 1/2 * t^2 - t - 1 \\ -e^t - 1/2 * t - 3/4 \end{pmatrix}$$

(c) (5 points) Show that there are no constant solutions to the non-homogeneous system above such that $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Solution. If there is such constant solution, then the following equation must hold.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ e^t \\ e^t \end{pmatrix}.$$

However, it is easy to see that the above equation cannot hold for any t.

2. (20 points) Let A be the matrix $A = \begin{pmatrix} 16 & -35 \\ 6 & -13 \end{pmatrix}$.

(a) (5 points) Compute the matrix exponential e^{At}.

Solution. Compute eigenvalues $1, 2$ and eigenvectors $(7,3)'$ and $(5,2)'$

$$A = \begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 3 & -7 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 3 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -14e^t + 15e^{2t} & 35e^t - 35e^{2t} \\ -6e^t + 6e^{2t} & 15e^t - 14e^{2t} \end{pmatrix}$$

(b) (10 points) Find the unique solution $\vec{x}(t)$ of the initial value problem

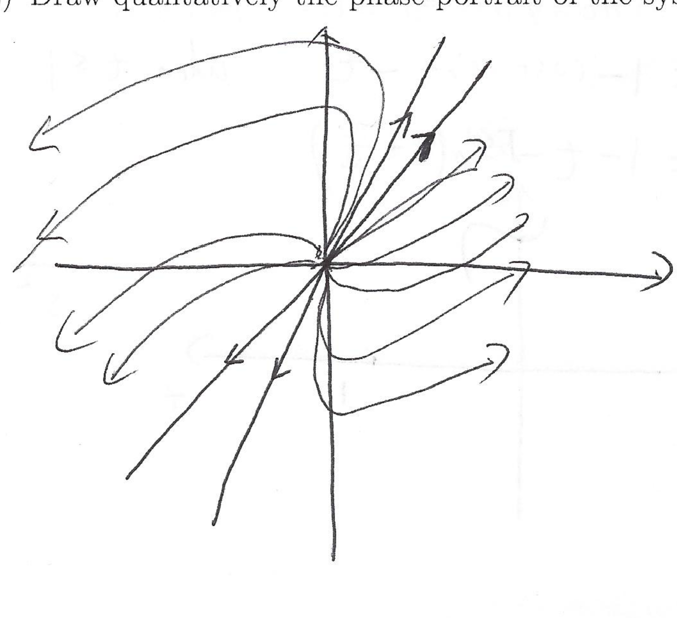
$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 16 & -35 \\ 6 & -13 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$e^{At}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

plug in and solve $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow C_1 = C_2 = 1$

$$\vec{X}(t) = \begin{pmatrix} 21e^t - 20e^{2t} \\ 9e^t - 8e^{2t} \end{pmatrix}$$

(c) (5 points) Draw qualitatively the phase-portrait of the system in Part (b).



3. (20 points) Consider the following Initial Value Problem (IVP):

$$y''(t) + y(t) = (1 - t) - u_1(t)(1 - t), \quad y(0) = 1, \quad y'(0) = 0,$$

where $u_1(t)$ is the Heaviside step function centered at $t = 1$.

(a) (5 points) Find the Laplace transform of the above differential equation and isolate $\mathcal{L}(y(t))(s)$ as a function of s.

$$s^2 L(y) + s\,y(0) + y'(0) + L(y) = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}$$

$$(s^2+1)\,L(y) + s = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2} = \frac{e^{-s} + s - 1}{s^2}$$

$$L(y) = \frac{s - s^3 + e^{-s} - 1}{s^2(s^2+1)}$$

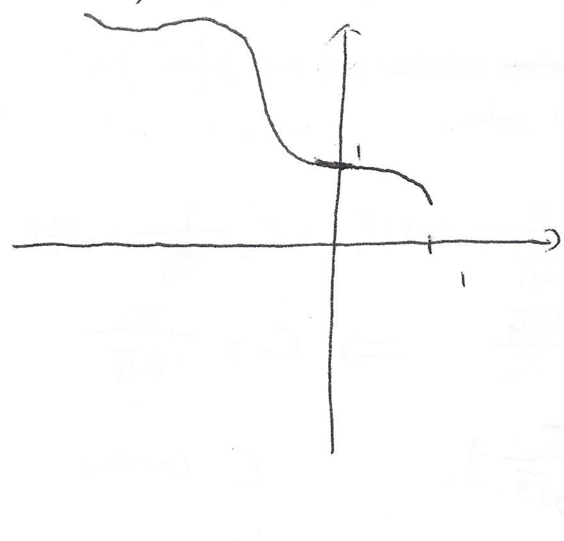
(b) (10 points) Find the unique solution $y(t)$ to the above Initial Value Problem.

$$L(y) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2+1} + \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s^2+1}$$

$$y = 1 - t + \sin t + (t-1)u_1(t) - \sin(t-1)\,u_1(t)$$

(c) (5 points) Plot qualitatively the solution $y(t)$ to the IVP above for $t \in (-\infty, 1)$. (*Hint: The system reads* $y''(t) + y(t) = (1 - t)$ *for* $t \leq 1$)

why $t \leq 1$, $y = 1 - t + \sin t$



4. (20 points) Consider the linear differential equation:

$$t^2y''(t) + 3ty'(t) + 4y(t) = 0, \quad t > 0.$$

(a) (5 points) Show that the two functions

$$y_1(t) = 12t^{-1}\cos(\sqrt{3}\ln(t)), \quad y_2(t) = 9t^{-1}\sin(\sqrt{3}\ln(t))$$

are solutions to the differential equation above.

$y_1'(t) = \left[-12\sqrt{3}\sin(\sqrt{3}\ln t) - 12\cos(\sqrt{3}\ln t)\right]/t^2$

$y_1''(t) = \left[-12\cos(\sqrt{3}\ln t) + 36\sqrt{3}\sin(\sqrt{3}\ln t)\right]/t^3$

$y_2'(t) = \left[9\sqrt{3}\cos(\sqrt{3}\ln t) - 9\sin(\sqrt{3}\ln t)\right]/t^2$

$y_2''(t) = \left[-9\sin(\sqrt{3}\ln t) - 27\sqrt{3}\cos(\sqrt{3}\ln t)\right]/t^3$

plug in and verify.

(b) (5 points) Compute the Wronskian of the solutions $y_1(t)$ and $y_2(t)$.

$$W = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' = \frac{108\sqrt{3}\cos^2\ast - 108\sin\ast\cos\ast}{t^3} - \frac{-108\sqrt{3}\sin^2\ast - 108\sin\ast\cos\ast}{t^3}$$

$\ast = \sqrt{3}\ln t$

$$= \frac{108\sqrt{3}}{t^3}$$

(c) (5 points) Find the *all* solutions to the differential equation above.

Since Wronskian is not 0, then y_1, y_2 are linearly independent.

Notice that ~~[illegible]~~ any linear combination of y_1, y_2 is also a solution.

Therefore general solution

$$y = C_1 y_1 + C_2 y_2$$

(d) (5 points) Find the ~~unique~~ solution $y(t)$ with $y\left(e^{\frac{\pi}{2\sqrt{3}}}\right) = 1$.

plug in the initial value.

$$1 = C_1 \frac{12}{\frac{\pi}{2\sqrt{3}}}\cos\left(\frac{\pi}{2}\right) + C_2\frac{9}{\frac{\pi}{2\sqrt{3}}}\sin\left(\frac{\pi}{2}\right)$$

$$= C_2 \cdot \frac{18\sqrt{3}}{\pi} \Rightarrow C_2 = \frac{\pi}{18\sqrt{3}} \qquad C_1 \text{ is free}$$

$$y = Cy_1 + \frac{\pi}{18\sqrt{3}}y_2 \qquad C: \text{Constant.}$$

5. (20 points) For each of the sentences below, circle the unique correct answer. By definition, the option "Neither" is the correct answer if the other options are not correct.

(a) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$ is:

(1) $\begin{pmatrix} 0 & e^t \\ 0 & 0 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ (3) $\begin{pmatrix} e^t & t \\ 0 & e^t \end{pmatrix}$ (4) $\begin{pmatrix} 1 & e^t \\ 0 & 1 \end{pmatrix}$

(b) (2 points) The phase-portrait of the system

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 22313 \\ -132193 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

can be qualitatively depicted as:

(1) Concentric circles (2) Inwards Spiral (3) Outwards Spiral (4) Neither

(c) (2 points) An autonomous differential equation will always have a:

(1) Stable solution (2) Unstable solution (3) Semistable solution (4) Neither

(d) (2 points) The global error in Euler's method with $h = 0.001$ is of order:

(1) 10^{-1} (2) 10^{-2} (3) 10^{-3} (4) 10^{-4}

(e) (2 points) The following function $y(t)$ solves the ODE $y'(t) + y^2(t) = 2t^{-2}$:

(1) $t + 1$ (2) $(2t^3 - 1)t^{-1}(1 + t^3)^{-1}$ (3) $\exp(t) + t^2$ (4) Neither.