University of California Davis **Name (Print):** ____________________

Differential Equations MAT 22B **Student ID (Print):** ____________________

Sample Final Examination III **December 9 2019**

Time Limit: 120 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following Initial Value Problem:

$$y''(t) + 4y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the function depicted in Figure

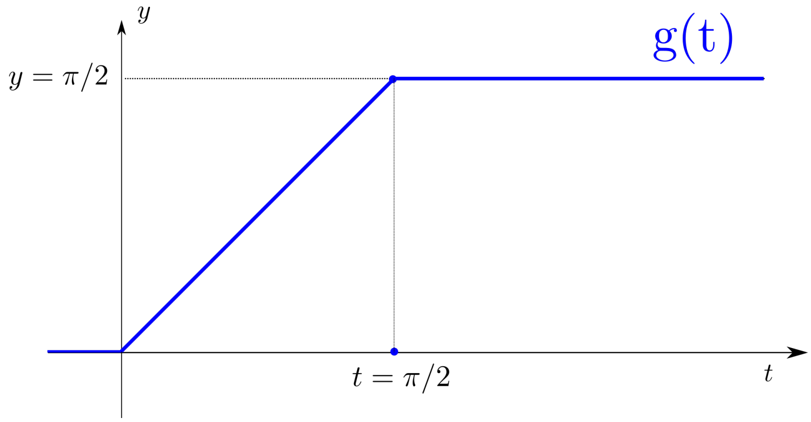

Figure 1: The external force $g(t)$.

(a) (5 points) Find the constant $A \in \mathbb{R}$ such that $g(t) = t - u_{\pi/2}(t)(t - A)$.

(b) (5 points) Calculate the Laplace transform of the Initial Value Problem above.

(c) (10 points) Find the unique solution $y(t)$ of the Initial Value Problem above.

2. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

(a) (5 points) Show that this system has *no* non-zero constant solutions.

(b) (10 points) Find *all* solutions to the system of differential equations.

(c) (5 points) Let $l \subseteq \mathbb{R}^3$ be the line spanned by the vector $(-2, 0, 1)^t$ and $w \in l$ any spanning vector. Explain why a solution $\vec{x}(t)$ to the above system with initial condition $\vec{x}(0) = w$ will stay inside the line l for all time, i.e. $\vec{x}(t) \in l$ for all $t \in \mathbb{R}$.

3. (20 points) Let $\alpha \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (5 points) Find the interval of values for $\alpha \in \mathbb{R}$ such that the linear system above admits non-zero solutions whose long-term behavior is zero.

(b) (5 points) For which values of the $\alpha \in \mathbb{R}$ does the above linear system have a non-zero constant solution ?

(c) (5 points) Plot qualitatively the phase-portrait of the system for $\alpha \in (10, \infty)$.

(d) (5 points) Plot qualitatively the phase-portrait of the system for $\alpha = i$, i.e. α is the purely imaginary number i.

4. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -1 + 21t + e^{3t}\tan(t) \\ 2 + 30t + 2e^{3t}\tan(t) \end{pmatrix}.$$

(a) (5 points) Find a fundamental matrix $\Psi(t)$ for the *homogeneous* problem associated to linear system above.

(b) (10 points) Find a particular solution $\vec{x}_p(t)$ to the initial non-homogeneous system. (*Hint*: an integral of $\tan(t)$ is $-\ln(\cos(t))$.)

(c) (5 points) Find the unique solution $\vec{x}(t) = (x_1(t), x_2(t))^t$ to the non-homogeneous system above which satisfies the initial conditions

$$x_1(1) = e, \qquad x_2(0) = 0.$$

5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (2 points) There are no constant solutions to $y'''(t) - t^3 y''(t) + \sin(y'(t)) + y(t) = e^t$.

(1) True. (2) False.

(b) (2 points) The Laplace transform of a constant function is a constant function.

(1) True. (2) False.

(c) (2 points) The Laplace transform of $\mathcal{L}(f(t)^2)(s)$ is $(\mathcal{L}(f(t))(s))^2$.

(1) True. (2) False.

(d) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}$.

(1) True. (2) False.

(e) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$ is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(1) True. (2) False.

(f) (2 points) The autonomous differential equation $y'(t) = \cos(t) + 1$ has infinitely many semistable solutions:

(1) True. (2) False.

(g) (2 points) The autonomous differential equation $y'(t) = \cos(t)$ has infinitely many semistable solutions:

(1) True. (2) False.

(h) (2 points) The long-term behavior of any solution $y(t)$ of the differential equation $y''(t) - 6y'(t) + 9y(t) = 0$ must always be zero.

(1) True. (2) False.

(i) (2 points) Suppose that A is a square matrix with distinct eigenvalues. Then the only constant solution of $\vec{x}(t)' = A\vec{x}(t)$ is zero.

(1) True. (2) False.

(j) (2 points) Let A be a 2×2 square matrix with $\det(A) \neq 0$ and $\vec{b} \in \mathbb{R}^2$ a vector. Then the system $\vec{x}(t)' = A\vec{x}(t) + \vec{b}$ always has a non-zero constant solution.

(1) True. (2) False.