University of California Davis **Name (Print):** ____________________
Differential Equations MAT 22B **Student ID (Print):** ____________________

Sample Final Examination III **December 9 2019**
Time Limit: 120 Minutes

This examination document contains 8 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following Initial Value Problem:

$$y''(t) + 4y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad t \geq 0,$$

where $g(t)$ is the function depicted in Figure

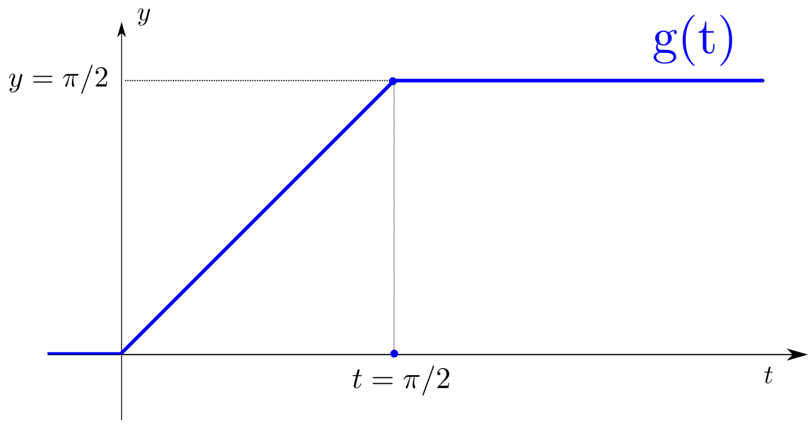

Figure 1: The external force $g(t)$.

(a) (5 points) Find the constant $A \in \mathbb{R}$ such that $g(t) = t - u_{\pi/2}(t)(t - A)$.

$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t < \frac{\pi}{2} \\ \frac{\pi}{2} & \text{if } t \geq \frac{\pi}{2} \end{cases}$$

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Then from this we get that

$$g(t) = t - u_{\frac{\pi}{2}}(t) \cdot (t - \frac{\pi}{2}) \tag{1}$$

(b) (5 points) Calculate the Laplace transform of the Initial Value Problem above.

$$y''(t) + 4y(t) = t - u_{\frac{\pi}{2}}(t) \cdot (t - \frac{\pi}{2})$$

from this equation we take the Laplace transform

$$\mathscr{L}\{y''(t) + 4y(t)\} = \mathscr{L}\{t - u_{\frac{\pi}{2}}(t) \cdot (t - \frac{\pi}{2})\}$$

when simplifying you finalize having

$$y(s) = \frac{1}{4s^2} - \frac{1}{s^2+4} + \frac{e^{\frac{-\pi}{2}s}}{4(s^2+4)} - \frac{e^{\frac{-\pi}{2}s}}{4s^2}$$

(c) (10 points) Find the unique solution $y(t)$ of the Initial Value Problem above. To find this final part you will need to take the inverse of the part b this results in the answer.

$$y(t) = \frac{t}{4} - \frac{1}{8}\sin 2t + u_{\frac{\pi}{2}}(t)(\frac{1}{8}\sin 2t - \pi - \frac{1}{4}(t - \frac{\pi}{2})$$

2. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

(a) (5 points) Show that this system has *no* non-zero constant solutions.

Since $\det(A) = 1 \neq 0$, there is no nullspace for A, and thus no non-zero constant solution exists.

(b) (10 points) Find *all* solutions to the system of differential equations.

The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 1$ with multiplicity three and defective. The only actual eigenvector $\xi_1^{(1)}$ is

$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

and the two generalized eigenvectors $\xi_1^{(2)}, \xi_1^{(3)}$ are

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) (5 points) Let $l \subseteq \mathbb{R}^3$ be the line spanned by the vector $(-2, 0, 1)^t$ and $w \in l$ any spanning vector. Explain why a solution $\vec{x}(t)$ to the above system with initial condition $\vec{x}(0) = w$ will stay inside the line l for all time, i.e. $\vec{x}(t) \in l$ for all $t \in \mathbb{R}$.

If a solution starts at an eigenline, it stays at the eigenline.

3. (20 points) Let $\alpha \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (5 points) Find the interval of values for $\alpha \in \mathbb{R}$ such that the linear system above admits non-zero solutions whose long-term behavior is zero.

What this question is asking for what interval for a which you get at least one of the eigenvalue being real and negative. Thus we get

$$-\infty < \alpha < 1$$

(b) (5 points) For which values of the $\alpha \in \mathbb{R}$ does the above linear system have a non-zero constant solution ?

This only happens when the determinant for the matrix is equal to zero so $\alpha = 1$

(c) (5 points) Plot qualitatively the phase-portrait of the system for $\alpha \in (10, \infty)$.

For this we know since that $\alpha \neq 1$ then the only constant solution is when (0,0) hence the origin is a fixed points. From here we know since both the trace and determinant are positive meaning both eigenvalues are negative so regardless of any intial conditions they will all converge to the origin.

(d) (5 points) Plot qualitatively the phase-portrait of the system for $\alpha = i$, i.e. α is the purely imaginary number i.

From this we get that both of the eigenvalues for this matrix will be both imaginary meaning that you get a unstable spiral on the origin.

4. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -1 + 21t + e^{3t}\tan(t) \\ 2 + 30t + 2e^{3t}\tan(t) \end{pmatrix}.$$

(a) (5 points) Find a fundamental matrix $\Psi(t)$ for the *homogeneous* problem associated to linear system above.
to find the fundamental matrix we solve for the eigenvalues of the matrix.

$$\lambda^2 - 2\lambda - 3 = 0, \lambda = -1, \lambda = 3$$

from this we solve for the eigen vectors and get the fundamental matrix

$$\Psi(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$$

(b) (10 points) Find a particular solution $\vec{x}_p(t)$ to the initial non-homogeneous system. (*Hint*: an integral of $\tan(t)$ is $-\ln(\cos(t))$.)
From the readings you should know that

$$\psi \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = g(t)$$

where g(t) is the part that makes the equation homogeneous. From here you will solve using row reductions learned from 22A getting the equations

$$u_1'(t) = -e^t + 3te^t$$

$$u_2' = 18te^{-3t} + \tan t$$

take the integral of both and you get your practical solution.

$$\begin{pmatrix} 3te^t + 4e^t \\ -\ln\cos t - 6te^{-3t} - 2e^{-3t} \end{pmatrix}$$

(c) (5 points) Find the unique solution $\vec{x}(t) = (x_1(t), x_2(t))^t$ to the non-homogeneous system above which satisfies the initial conditions

$$x_1(1) = e, \qquad x_2(0) = 0.$$

Using Part b we can solve for the IVP by solving using

$$\mathbf{x(t)} = \Psi(t)\mathbf{u(t)}$$

Solving this using basic matrix multiplication you get the equation

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -3t - 6 + c_1e^{-t} - e^{3t}\ln\cos t + c_2e^{3t} \\ -18t + 4 + c_1e^{-t} - 2e^{3t}\ln\cos t + c_2e^{3t} \end{pmatrix}.$$

Then you finally solve using the the initial values where you get

$$c_2 = \frac{e + 9 - e^3\ln\cos 1 - 6e^{-}}{e^3 - e^{-t}}$$

and $c_1 = 6 - c_2$.

5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (2 points) There are no constant solutions to $y'''(t) - t^3y''(t) + \sin(y'(t)) + y(t) = e^t$.
(1) True. (2) False.
This is TRUE, if you get y(t)=c where is a constant then all derivatives will be zero. In the end you will be left with

$$c = e^t$$

which is not a true statement since c is a constant not an equation.

(b) (2 points) The Laplace transform of a constant function is a constant function.

(1) True. (2) False.
This is FALSE, if you take the laplace transform of a constant function is actually

$$f(s) = \frac{c}{s}$$

function of s

(c) (2 points) The Laplace transform of $\mathcal{L}(f(t)^2)(s)$ is $(\mathcal{L}(f(t))(s))^2$.

(1) True. (2) **False**.

(d) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}$.

(1) True. (2) False.
This is TRUE

(e) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$ is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(1) True. (2) False.

This is TRUE

(f) (2 points) The autonomous differential equation $y'(t) = \cos(y(t)) + 1$ has infinitely many semistable solutions:

(1) True. (2) False.

This is true.

(g) (2 points) The autonomous differential equation $y'(t) = \cos(y(t))$ has infinitely many semistable solutions:

(1) True. (2) **False**.

(h) (2 points) The long-term behavior of any solution $y(t)$ of the differential equation $y''(t) - 6y'(t) + 9y(t) = 0$ must always be zero.

(1) True. (2) False.

This is a false equation if you take the characteristic equation we see that both of the λ are positive hence they are growing exponentially.

(i) (2 points) Suppose that A is a square matrix with distinct eigenvalues. Then the only constant solution of $\vec{x}(t)' = A\vec{x}(t)$ is zero.

(1) True. (2) **False**.

(j) (2 points) Let A be a 2×2 square matrix with $\det(A) \neq 0$ and $\vec{b} \in \mathbb{R}^2$ a vector. Then the system $\vec{x}(t)' = A\vec{x}(t) + \vec{b}$ always has a non-zero constant solution.

(1) True. (2) False.

This is a FALSE because b might be zero.