This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (20 points) Consider the following linear system of differential equations:
\[
\begin{pmatrix}
  x_1'(t) \\
  x_2'(t) \\
  x_3'(t)
\end{pmatrix} =
\begin{pmatrix}
  -1 & 1 & 5 \\
  0 & -1 & 4 \\
  0 & 0 & -5
\end{pmatrix}
\begin{pmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t)
\end{pmatrix}.
\]

(a) (5 points) Find a fundamental set of solutions to the differential system.
The eigenvalues are: \( \lambda_1 = \lambda_2 = -1, \lambda_3 = -5 \). Since \( \lambda_1 = -1 \) is of multiplicity 2, there is one eigenvector \( v_1 \) and one generalized eigenvector \( v_2 \):
\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]
The eigenvector associated with \( \lambda_3 = -5 \) is \( v_3 = [1, 1, -1]^T \). Hence the fundamental set is
\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-5t} \right\}.
\]

(b) (5 points) Find all solutions to the system of differential equations above.
Superposition of 3 fundamental solution gives all solutions.

(c) (5 points) Find the solutions to the system which satisfy
\[
\begin{pmatrix}
  x_1(0) \\
  x_2(0) \\
  x_3(0)
\end{pmatrix} =
\begin{pmatrix}
  12 \\
  0 \\
  1
\end{pmatrix}.
\]
Let 3 underdetermined constants in the general solution be \( c_1, c_2, c_3 \). Plug-in the initial condition and we have constants \( c_1 = 13, c_2 = 1, c_3 = -1 \).
\[
X(t) = 13 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-5t}.
\]

(d) (5 points) Compute the long-term behavior of all solutions \( \vec{x}(t) \).
All solutions will converge to 0 since no eigenvalue is greater than 0.
2. (20 points) Let $\gamma \in \mathbb{R}$ and consider the following system of differential equations:

$$
\begin{pmatrix}
  x'_1(t) \\
  x'_2(t)
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  -1 & \gamma
\end{pmatrix} \begin{pmatrix}
  x_1(t) \\
  x_2(t)
\end{pmatrix}.
$$

(a) (5 points) Draw the phase-portrait of the system for the three values $\gamma \in \{-1, 0, 1\}$.

The shape of portraits are listed as follows:

- $\gamma = -1$, clockwisely spinning in approaching 0.
- $\gamma = 0$, concentric circles.
- $\gamma = 1$, clockwisely spinning out from 0.

(b) (5 points) Set $\gamma = 3$, compute the long-term behaviour of the unique solution $\vec{x}(t)$ with the initial condition $x(0) = \begin{pmatrix} 0.002 \\ -0.03 \end{pmatrix}$.

Eigenvalues and eigenvectors are:

$$
\begin{align*}
\lambda_1 &= (3 - \sqrt{5})/2 \\
\lambda_2 &= (3 + \sqrt{5})/2 \\
v_1 &= [(3 + \sqrt{5})/2, 1]^\top \\
v_2 &= [(3 - \sqrt{5})/2, 1]^\top.
\end{align*}
$$

Since both eigenvalues are greater than 0, and since the given initial condition lies in the negative direction of both eigenvectors, the long term behaviour is diverging to $[-\infty, -\infty]^\top$.

(c) (5 points) Is there a value of $\gamma \in \mathbb{R}$ such that the phase-portrait is not a spiral but the long-term behaviour of all solutions is still zero?

This happens when 2 eigenvalues are real and negative, or one negative one 0. We have for any $\gamma$:

$$
\lambda = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2},
$$

hence for $\gamma \leq -2$, every solution converges to 0 and they are not spirals since the eigenvalues are real.

(d) (5 points) Find all the values $\gamma \in \mathbb{R}$ for which phase-portrait of the system behaves as a spiral. (Concentric circles are considered spirals as well.)

This happens when $\lambda$ is complex, hence we have $\lambda \in (-2, 2)$. 

3. (20 points) Consider the following differential equation:

\[ y''(t) + y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

(a) (10 points) Find the unique solution to the above differential equation for the external force \( g(t) = \delta(t - 2) + \delta(t - 15) \) and plot it.

Take the Laplace transform:

\[
\begin{align*}
    s^2 L(y) - sy'(0) - y(0) + L(y) &= L(g) \\
    (s^2 + 1)L(y) &= e^{-2s} + e^{-15s} \\
    L(y) &= \frac{1}{s^2 + 1} (e^{-2s} + e^{-15s}).
\end{align*}
\]

Hence the solution is

\[ y(t) = \sin(t - 2)u(t - 2) + \sin(t - 15)u(t - 15), \]

where \( u(t) \) is the Heaviside step function.

(b) (5 points) Find the unique solution to the above differential equation for the external force \( g(t) = \delta(t - \pi) + \delta(t - 2\pi) \). How does its long-term behavior differ from the solution in Part (a) ?

\[ y(t) = \sin(t - \pi)u(t - \pi) + \sin(t - 2\pi)u(2 - 2\pi) = -\sin(t)u(t - \pi) + \sin(t)u(t - 2\pi). \]

\( y(t) \) is zero when \( t > 2\pi \).

(c) (5 points) Find an external force \( g(t) \) such that the above Initial Value Problem has a solution which qualitatively looks as in Figure 1.

The solution is \(-\sin(t)\) when \( t \in [\pi, 2\pi] \cup [11\pi, 12\pi] \) and zero everywhere else. Hence we can write \( y(t) \) in terms of Heaviside function times some sine function:

\[
\begin{align*}
    y(t) &= -\sin(t)u(t - \pi) + \sin(t)u(t - 2\pi) - \sin(t)u(t - 11\pi) + \sin(t)u(t - 12\pi) \\
    &= \sin(t - \pi)u(t - \pi) + \sin(t - 2\pi)u(t - 2\pi) \\
    &\quad + \sin(t - 11\pi)u(t - 11\pi) + \sin(t - 12\pi)u(t - 12\pi).
\end{align*}
\]

The inverse transform gives:

\[ g(t) = \delta(t - \pi) + \delta(t - 2\pi) + \delta(t - 11\pi) + \delta(t - 12\pi). \]
4. (20 points) Consider the non-homogeneous linear system of differential equations:

\[
\begin{pmatrix}
  x_1'(t) \\
  x_2'(t)
\end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\
  x_2(t)
\end{pmatrix} + \begin{pmatrix} 2e^{4t} \\
  e^{4t}
\end{pmatrix}.
\]

(a) (10 points) Find a particular solution \( \vec{x}_p(t) \) to the linear system above. Guess the particular solution as \( X(t) = [a, b]^T e^{4t} \). Plug-in and we have

\[
4 \begin{pmatrix} a \\
  b
\end{pmatrix} e^{4t} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\
  b
\end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\
  1
\end{pmatrix} e^{4t}.
\]

Cancel \( e^{4t} \) and the linear system remaining gives

\[
a = \frac{8}{5}, \quad b = \frac{7}{5}.
\]

(b) (5 points) Find all solutions to the linear system above.

The matrix has eigenvalues and eigenvectors:

\[
\lambda_1 = -1 \quad \lambda_2 = 3 \\
v_1 = [-1, 1]^T \quad v_2 = [1, 1]^T.
\]

Hence the general solution is given by:

\[
X(t) = c_1 \begin{pmatrix} -1 \\
  1
\end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\
  1
\end{pmatrix} e^{3t} + \begin{pmatrix} 8/5 \\
  7/5
\end{pmatrix} e^{4t},
\]

(c) (5 points) Find all solutions with \( x_1(0) = \frac{8}{5} \).

Plug-in the initial condition gives \( c_1 = c_2 \).
5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (2 points) The exponential of the matrix \( \begin{pmatrix} 2t & 0 \\ 0 & -4t \end{pmatrix} \) is \( \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-4t} \end{pmatrix} \).

(1) True. (2) False.

The matrix is diagonal.

(b) (2 points) The exponential of the matrix \( \begin{pmatrix} t & t \\ 0 & t \end{pmatrix} \) is \( \begin{pmatrix} e^t & e^t \\ 0 & e^t \end{pmatrix} \).

(1) True. (2) False.

The matrix is not diagonal. One should compute the eigendecomposition first.

(c) (2 points) There exist a autonomous first-order differential equation with infinitely many semistable solutions.

(1) True. (2) False.

For example, take \( f(x) = \sin(x) + 1 \), and \( x'(t) = f(x) \).

(d) (2 points) If an autonomous first-order differential equation has two stable solutions then it must have a unstable solution.

(1) True. (2) False.

One needs to go across 0 3 times to have two stable nodes, and the rest the one unstable node.

(e) (2 points) The local error in Euler’s method with step \( h = 0.01 \) is of order \( 10^{-1} \):

(1) True. (2) False.

The order of local error is \( O(h^2) \).

(f) (2 points) A homogeneous linear system of differential equations \( \ddot{x}(t) = Ax(t) \) never has a non-zero constant solution.

(1) True. (2) False.

It depends on the system. There is a non-zero constant solution if \( \det(A) = 0 \).

(g) (2 points) A linear system of differential equations \( \ddot{x}(t) = A\ddot{x}(t) + g(t) \) might have non-zero constant solutions for certain \( g(t) \).

(1) True. (2) False.

Pick \( g \) to be a constance vector and \( A \) is a full rank matrix will do the trick.

(h) (2 points) The Laplace transform \( \mathcal{L}(f)(s) \) of a differentiable function \( f : \mathbb{R} \rightarrow \mathbb{R} \) always exist and is differentiable with respect to \( s \).

(1) True. (2) False.

One needs \( f(t)e^{-st} \) integrable therefore \( f(t) \) cannot grow faster than \( e^t \).
(i) (2 points) The Laplace transform $L$ is a linear transformation.

(1) True. (2) False.

(j) (2 points) The non-linear system of two differential equations

\[ x'(t) = y(t) - x^3(t) + x(t), \quad y'(t) = -x(t)e^{y(t)} \]

has exactly two constant solutions.

(1) True. (2) False.

Set

\[ y(t) - x^3(t) - x(t) = 0 \]
\[ -x(t)\exp(y(t)) = 0. \]

And we see that $x = 0, y = 0$ is the only solution.