**University of California Davis** **Name (Print):** ____________________

**Differential Equations MAT 22B** **Student ID (Print):** ____________________

**Sample Final Examination** **December 9 2019**

**Time Limit: 120 Minutes**

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This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total: | 100 | |

1. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & 5 \\ 0 & -1 & 4 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

(a) (5 points) Find a fundamental set of solutions to the differential system.

The eigenvalues are: $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = -5$. Since $\lambda_1 = -1$ is of multiplicity 2, there is one eigenvector $v_1$ and one generalized eigenvector $v_2$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The eigenvector associated with $\lambda_3 = -5$ is $v_3 = [1, 1, -1]^\top$. Hence the fundamental set is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}, \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{-t}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-5t} \right\}.$$

(b) (5 points) Find *all* solutions to the system of differential equations above.

Superposition of 3 fundamental solution gives all solutions.

(c) (5 points) Find the solutions to the system which satisfy $\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix}$.

Let 3 underdetermined constants in the general solution be $c_1, c_2, c_3$. Plug-in the initial condition and we have constants $c_1 = 13, c_2 = 1, c_3 = -1$.

$$X(t) = 13 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{-t} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-5t}.$$

(d) (5 points) Compute the long-term behavior of *all* solutions $\vec{x}(t)$.

All solutions will converge to 0 since no eigenvalue is greater than 0.

2. (20 points) Let $\gamma \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \gamma \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (5 points) Draw the phase-portrait of the system for the three values $\gamma \in \{-1, 0, 1\}$.

The shape of portraits are listed as follows:

- $\gamma = -1$, clockwisely spinning in approaching 0.
- $\gamma = 0$, concentric circles.
- $\gamma = 1$, clockwisely spinning out from 0.

(b) (5 points) Set $\gamma = 3$, compute the long-term behaviour of the unique solution $\vec{x}(t)$ with the initial condition $x(0) = \begin{pmatrix} 0.002 \\ -0.03 \end{pmatrix}$.

Eigenvalues and eigenvectors are:

$$\begin{array}{ll} \lambda_1 = (3 - \sqrt{5})/2 & \lambda_2 = (3 + \sqrt{5})/2 \\ v_1 = [(3 + \sqrt{5})/2, 1]^\top & v_2 = [(3 - \sqrt{5})/2, 1]^\top. \end{array}$$

Since both eigenvalues are greater than 0, and since the given initial condition lies in the negative direction of both eigenvectors, the long term behaviour is diverging to $[-\infty, -\infty]^\top$.

(c) (5 points) Is there a value of $\gamma \in \mathbb{R}$ such that the phase-portrait is *not* a spiral but the long-term behaviour of *all* solutions is still zero ?

This happens when 2 eigenvalues are real and negative, or one negative one 0. We have for any $\gamma$:

$$\lambda = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2},$$

hence for $\gamma \leq -2$, every solution converges to 0 and they are not spirals since the eigenvalues are real.

(d) (5 points) Find all the values $\gamma \in \mathbb{R}$ for which phase-portrait of the system behaves as a spiral. (Concentric circles are considered spirals as well.)

This happens when $\lambda$ is complex, hence we have $\lambda \in (-2, 2)$.

3. (20 points) Consider the following differential equation:

$$y''(t) + y(t) = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

(a) (10 points) Find the unique solution to the above differential equation for the external force $g(t) = \delta(t-2) + \delta(t-15)$ and plot it.

Take the Laplace transform:

$$\begin{aligned} &s^2\mathcal{L}(y) - sy'(0) - y(0) + \mathcal{L}(y) = \mathcal{L}(g) \\ &(s^2+1)\mathcal{L}(y) = e^{-2s} + e^{-15s} \\ &\mathcal{L}(y) = \frac{1}{s^2+1}\left(e^{-2s} + e^{-15s}\right). \end{aligned}$$

Hence the solution is

$$y(t) = \sin(t-2)u(t-2) + \sin(t-15)u(t-15),$$

where $u(t)$ is the Heaviside step function.

(b) (5 points) Find the unique solution to the above differential equation for the external force $g(t) = \delta(t-\pi) + \delta(t-2\pi)$. How does its long-term behavior differ from the solution in Part (a) ?

$$\begin{aligned} y(t) &= \sin(t-\pi)u(t-\pi) + \sin(t-2\pi)u(2-2\pi) \\ &= -\sin(t)u(t-\pi) + \sin(t)u(t-2\pi). \end{aligned}$$

$y(t)$ is zero when $t > 2\pi$.

(c) (5 points) Find an external force $g(t)$ such that the above Initial Value Problem has a solution which qualitatively looks as in Figure 1.

The solution is $-\sin(t)$ when $t \in [\pi, 2\pi] \cup [11\pi, 12\pi]$ and zero everywhere else. Hence we can write $y(t)$ in terms of Heviside function times some sine function:

$$\begin{aligned} y(t) &= -\sin(t)u(t-\pi) + \sin(t)u(t-2\pi) - \sin(t)u(t-11\pi) + \sin(t)u(t-12\pi) \\ &= \sin(t-\pi)u(t-\pi) + \sin(t-2\pi)u(t-2\pi) \\ &+ \sin(t-11\pi)u(t-11\pi) + \sin(t-12\pi)u(t-12\pi). \end{aligned}$$

The inverse transform gives:

$$g(t) = \delta(t-\pi) + \delta(t-2\pi) + \delta(t-11\pi) + \delta(t-12\pi).$$

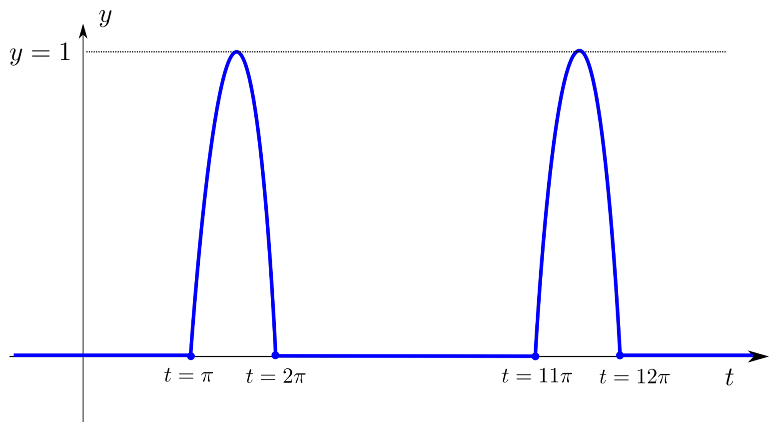


Figure 1: The plot of the solution for Part (c).

4. (20 points) Consider the non-homogeneous linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}.$$

(a) (10 points) Find a particular solution $\vec{x}_p(t)$ to the linear system above. Guess the particular solution as $X(t) = [a, b]^\top e^{4t}$. Plug-in and we have

$$4 \begin{bmatrix} a \\ b \end{bmatrix} e^{4t} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} e^{4t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

Cancel $e^{4t}$ and the linear system remaining gives

$$a = 8/5, \quad b = 7/5.$$

(b) (5 points) Find *all* solutions to the linear system above.

The matrix has eigenvalues and eigenvectors:

$$\begin{array}{ll} \lambda_1 = -1 & \lambda_2 = 3 \\ v_1 = [-1, 1]^\top & v_2 = [1, 1]^\top. \end{array}$$

Hence the general solution is given by:

$$X(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 8/5 \\ 7/5 \end{bmatrix} e^{4t},$$

(c) (5 points) Find *all* solutions with $x_1(0) = 8/5$.
Plug-in the initial condition gives $c_1 = c_2$.

5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (2 points) The exponential of the matrix $\begin{pmatrix} 2t & 0 \\ 0 & -4t \end{pmatrix}$ is $\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-4t} \end{pmatrix}$.

(1) True. (2) False.

The matrix is diagonal.

(b) (2 points) The exponential of the matrix $\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$ is $\begin{pmatrix} e^{t} & e^{t} \\ 0 & e^{t} \end{pmatrix}$.

(1) True. (2) False.

The matrix is not diagonal. One should compute the eigendecomposition first.

(c) (2 points) There exist a autonomous first-order differential equation with infinitely many semistable solutions.

(1) True. (2) False.

For example, take $f(x) = \sin(x) + 1$, and $x'(t) = f(x)$.

(d) (2 points) If an autonomous first-order differential equation has two stable solutions then it must have a unstable solution.

(1) True. (2) False.

One needs to go across 0 3 times to have two stable nodes, and the rest the one unstable node.

(e) (2 points) The local error in Euler's method with step $h = 0.01$ is of order $10^{-1}$:

(1) True. (2) False.

The order of local erroris $O(h^2)$.

(f) (2 points) A homogeneous linear system of differential equations $\vec{x}(t)' = A\vec{x}(t)$ never has a non-zero constant solution.

(1) True. (2) False.

It depends on the system. There is a non-zero constant solution if $\det(A) = 0$.

(g) (2 points) A linear system of differential equations $\vec{x}(t)' = A \cdot \vec{x}(t) + g(t)$ might have non-zero constant solutions for certain $g(t)$.

(1) True. (2) False.

Pick $g$ to be a constance vector and $A$ is a full rank matrix will do the trick.

(h) (2 points) The Laplace transform $\mathcal{L}(f)(s)$ of a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ always exist and is differentiable with respect to $s$.

(1) True. (2) False.

One needs $f(t)e^{-st}$ integrable therefore $f(t)$ cannot grow faster than $e^t$.

(i) (2 points) The Laplace transform $\mathcal{L}$ is a linear transformation.

(1) True. (2) False.

Definition.

(j) (2 points) The non-linear system of two differential equations

$$x'(t) = y(t) - x^3(t) + x(t), \quad y'(t) = -x(t)e^{y(t)}$$

has exactly two constant solutions.

(1) True. (2) False.

Set

$$\begin{aligned} y(t) - x^3(t) - x(t) &= 0 \\ -x(t)\exp(y(t)) &= 0. \end{aligned}$$

And we see that $x = 0, y = 0$ is the only solution.