University of California Davis **Name (Print):** ________________
Differential Equations MAT 22B **Student ID (Print):** ________________

Sample Midterm Examination **October 25 2019**
Time Limit: 50 Minutes

This examination document contains 6 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following first-order differential equation:

$$y'(t) - 2y(t) = t.$$

(a) (10 points) Find *all* solutions to the differential equation.

Solution. Let us use the method of integrating factors. For that, multiply by the integrating factor $e^{\int -2dt} = e^{-2t}$ on both sides.

$$(e^{-2t}y(t))' = te^{-2t}$$

Take the integral on both sides.

$$e^{-2t}y = \int te^{-2t}dt = -\frac{2t+1}{4}e^{-2t} + C$$

$$y = Ce^{2t} - \frac{2t+1}{4}$$

(b) (10 points) Solve the Initial Value Problem given by $y(0) = 0$.

Solution. Plug in initial value and solve for the constant.

$$0 = C - \frac{1}{4}$$

$$C = \frac{1}{4}$$

2. (20 points) Consider the first-order differential equation $y'(t) = f(y(t))$ where $f(y)$ is depicted in Figure 1, and $f(y)$ decays to minus infinity away from the picture.

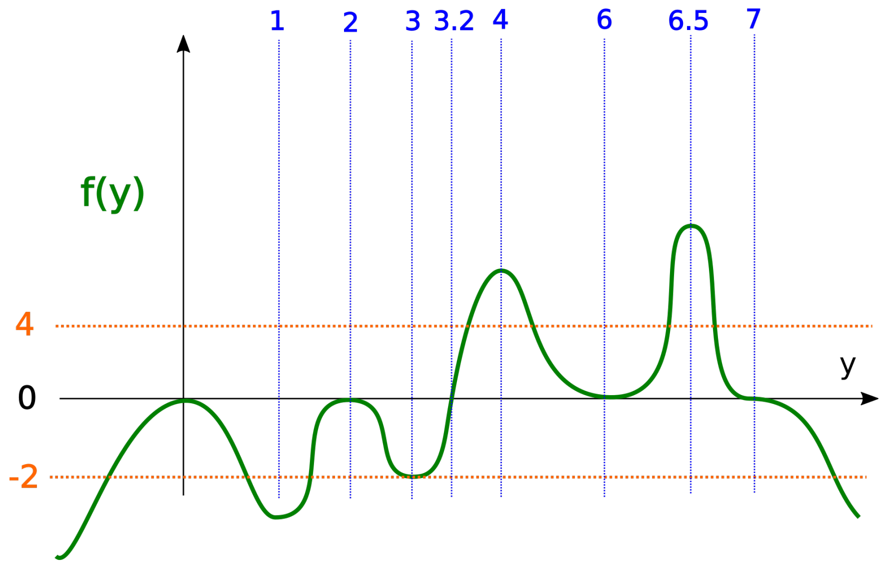

Figure 1: The function $f(y)$ for Problem 2.

(a) (5 points) How many constant solutions does $y'(t) = f(y(t))$ have ?

Solution. The graph has 5 intersections with the horizontal axis. There are 5 constant solutions.

$$y = 0, 2, 3.2, 6, 7.$$

(b) (5 points) Classify the constant solutions into *stable*, *unstable* and *semistable*.

0: semistable; 2: semistable: 3.2: unstable; 6: semistable; 7: stable.

(c) (5 points) Describe the long term behavior of the unique solution of $y'(t) = f(y(t))$ such that $y(0) = 4$?

Since $y(0) = 4$, the graph of this solution will lie between the constant solutions 3.2 and 6. Since 6 is stable and 3.2 is unstable, the long term behavior of y will increase and approach 6.

(d) (5 points) Find the number of constant solutions of $y'(t) = f(y(t)) + 2$.

Move the graph up by 2 units. There are 5 intersections with the horizontal axis, so still 5 constant solutions.

3. (20 points) Consider the following differential equation:

$$y''(t) + 4y'(t) + 4y(t) = 3t.$$

(a) (8 points) Find a particular solution to the above differential equation.

We solve by the Method of Undetermined Coefficients, and we suppose the solution is $y = At + B$. Then $y' = A$ and $y'' = 0$. Plug the solution into the equation.

$$0 + 4A + 4(At + B) = 3t$$

Compare the coefficients of linear term and constant term.

$$4A = 3, 4A + 4B = 0$$

Solve the equation and a particular solution is

$$y = \frac{3}{4}(t - 1)$$

(b) (8 points) Find *all* solutions to the above differential equation.

Solve the characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ and there is exactly one solution $\lambda = -2$. Then the two linearly independent solutions of $y''(t) + 4y'(t) + 4y(t) = 0$ are e^{-2t} and te^{-2t}.

$$y = C_1 e^{-2t} + C_2 te^{-2t} + \frac{3}{4}(t - 1)$$

(c) (4 points) Is the damped harmonic oscillator described by the differential equation *overdamped*, *critically damped* or *underdamped* ?

The characteristic equation has exactly one negative solution, i.e. a repeated root, so it is critically damped.

4. (20 points) Consider the second-order differential equation:

$$t^2 y''(t) - 3ty'(t) + 4y(t) = 0, \quad t > 0$$

(a) (5 points) Show that $y_1(t) = t^2$ is a solution to the differential equation above.

Compute the derivatives: $y_1'(t) = 2t$ and $y_1''(t) = 2$, and plug in to the ODE:

$$t^2 \cdot 2 - 3t \cdot 2t + 4 \cdot t^2 = 0.$$

(b) (5 points) Suppose that $v(t)y_1(t)$ is a solution to the differential equation above. Show that $v(t)$ satisfies

$$v''(t) + t^{-1}v'(t) = 0.$$

Suppose $y_2 = vy_1 = vt^2$ and $y_2' = v'y_1 + vy_1' = v't^2 + 2vt$ and $y_2'' = v''y_1 + 2v'y_1' + vy_1'' = v''t^2 + 4v't + 2v$. Plug in to the ODE.

$$0 = t^2 y''(t) - 3ty'(t) + 4y(t) = t^2(v''t^2 + 4v't + 2v) - 3t(v't^2 + 2vt) + 4vt^2.$$

After symplifying, we get

$$v''t^4 + v't^3 = 0$$

Since $t > 0$, we can divide t^4 and obtain $v''(t) + t^{-1}v'(t) = 0$.

(c) (5 points) Find a solution $y_2(t)$ which is linearly independent with $y_1(t)$.

Based on (b), we get $(tv')' = 0$. Then $tv' = C$ and $v' = \frac{C}{t}$. Finally $v = C_1 \ln t + C_2$. For simplicity, we can choose $C_1 = 1$ and $C_2 = 0$. The second solution is

$$y_2 = t^2 \ln t,$$

which has non-zero Wronskian with $y_1(t)$.

(d) (5 points) Find all solutions to the differential equation above.

Use linear combination of y_1, y_2, then we can find the all solutions.

$$y = C_1 t^2 + C_2 t^2 \cdot \ln t.$$

5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**.

(a) (2 points) The graphs of two solutions for $y'(t) + e^{t^9} y(t) = \cos(t^2)$ cannot intersect.

(1) True. (2) False.

(b) (2 points) The graphs of two solutions for $y''(t)+y'(t)+100y(t) = 2$ cannot intersect.

(1) True. (2) False.

(c) (2 points) Any autonomous first-order differential equation has a constant solution.

(1) True. (2) False.

(d) (2 points) The functions $y_1(t) = \sin(t)$ and $y_2(t) = \cos(t)$ have non-zero Wronskian.

(1) True. (2) False.

(e) (2 points) The local error in Euler's method with step h is of order $\mathcal{O}(h^2)$:

(1) True. (2) False.

(f) (2 points) Autonomous first-order differential equations must have finitely many constant solutions.

(1) True. (2) False.

(g) (2 points) The function $y(t) = e^{2t}$ solves the differential equation given by

$$y''(t) - 4y'(t) + 4y(t) = e^{2t}.$$

(1) True. (2) False.

(h) (2 points) The global error in Euler's method with step h is of order $\mathcal{O}(h)$.

(1) True. (2) False.

(i) (2 points) No solution $y_1(t)$ of $y'(t) = 2t$ satisfies $y_1(0) = 0$, $y_1(1) = 1$ *and* $y_1(3) = 9$.

(1) True. (2) False.

(j) (2 points) No solution $y_1(t)$ of $y'(t) = 2t$ satisfies $y_1(0) = 0$, $y_1(1) = 1$ *and* $y_1(2) = 3$.

(1) True. (2) False.