University of California Davis **Name (Print):** ____________________

Differential Equation MAT 22B **Student ID (Print):** ____________________

Sample Midterm Examination **October 25 2019**

Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following first-order differential equation:

$$y'(t) + 2ty(t) = t.$$

(a) (10 points) Find *all* solutions to the differential equation.

Solution: The differential equation can be rearranged to

$$y'(t) = (1 - 2y(t))t,$$

which yields a constant solution $y(t) = 1/2$ immediately. Then by separating variables we have

$$\frac{dy}{1 - 2y(t)} = tdt.$$

Integrating both sides gives:

$$\left(-\frac{1}{2}\right) \ln |1 - 2y(t)| = \frac{1}{2}t^2 + C.$$

This can be written as $|1 - 2y(t)| = Ce^{-t^2}$, where $C \in \mathbb{R}^+$. If $1 - 2y(t) > 0$ then this reads

$$y(t) = \frac{1}{2} - Ce^{-t^2},$$

and if $1 - 2y(t) < 0$ we obtain

$$y(t) = \frac{1}{2} + Ce^{-t^2},$$

where C is a positive constant. In conclusion, all solutions are of the form

$$y(t) = \frac{1}{2} + Ce^{-t^2},$$

where C is now an arbitrary real number, including zero, which corresponds to the constant solution.

(b) (5 points) Find *all* solutions to the differential equation which satisfy $y(0) = 1$.

Solution: By plugging the initial condition $y(0) = 1$ in the function $y(t)$ and solving for C we obtain:

$$1 = \frac{1}{2} + C.$$

Thus $C = 0.5$ and we have that

$$y(t) = 1/2 + 1/2e^{-t^2}$$

is the only solution satisfying $y(0) = 1$.

(c) (5 points) Compute the long-term behavior of *all* solutions $y(t)$.

Solution: The long-term behavior limit reads:

$$\lim_{t\to\infty}\left(\frac{1}{2}+Ce^{-t^2}\right)=\frac{1}{2}, \quad \forall C\in\mathbb{R}.$$

Therefore, all solutions will converge to $1/2$ in the long-term.

2. (20 points) Consider the following first-order differential equation:

$$y'(t)=y(1-2y)-6y+12, \quad t\geq 0.$$

(a) (10 points) Find the constant solutions of the above differential equation and classify them into *stable*, *unstable* and *semistable*.

Solution: Factorizing the right hand side gives:

$$y'(t)=(-2y+3)(y+4).$$

Therefore 2 constant solutions are:

$$y_1(t)=\frac{3}{2}, \;\; y_2(t)=-4.$$

By observing the sign changes of $y'(t)$ we know that $y_1(t)$ is stable while $y_2(t)$ is unstable.

(b) (5 points) Suppose that $y_1(t)$ is the unique solution of the above differential equation such that $y_1(5)=0$. Compute $\lim_{t\to\infty} y_1(t)$.

Solution: We know that

$$y_1'(5)=(0+3)(0+4)=12>0.$$

Therefore, since $y(t)=3/2$ is a stable equilibrium,

$$\lim_{t\to\infty} y_1(t)=3/2.$$

(c) (5 points) How many solutions $y(t)$ are there with $y(0)=0$ and $y(10)=2$?

Solution: Given that $3/2=1.5$ is a constant (stable) solution, it is not possible for *any* solution to have values both below *and* above 1.5, thus it is not possible to have any solutions with $y(0)=0$ and $y(10)=2$, since $0<3/2<2$. Alternatively, you can solve directly if you *really* want to. Indeed, solving this differential equation by separating variables:

$$\frac{dy}{(-2y+3)(y+4)} = dt$$
$$\left[\frac{2}{11(-2y+3)} + \frac{1}{11(y+4)}\right] dy = dt$$

Integrating both sides then we have:

$$y(t) = \frac{3e^{11t} - 4C}{2e^{11t} + C}.$$

Plug in $y(0) = 0$ gives $C = 4/3$ which does not satisfy $y(10) = 2$, hence there is no solution.

3. (20 points) Consider the following differential equation:

$$y'(t) = e^{t^2}.$$

In Part (c) you are allowed to leave the expression with terms such as e^2, $e^{-0.1}$, without having to evaluate them.

(a) (5 points) Show that no solution to the differential equation above converges to 0 in the limit $t \to \infty$. Is there any constant solution ?

Solution: Since $y'(t) \to \infty$ exponentially as $t \to \infty$, no matter what initial condition we have, the solution $y(t)$ will diverge. Thus there is no solution converging to 0 or being constant.

(b) (5 points) Is there a solution $y_1(t)$ with $y_1(0) = 4$ and $y_1(2) = 1$?

Solution: No there does not exist such solution because the given conditions satisfy $y_1(2) < y_1(0)$ while we know that any solution $y(t)$ should be increasing since $y'(t) = e^{t^2} > 0$.

(c) (10 points) Let $y_2(t)$ be the unique solution such that $y_2(0) = 0$. Approximate the value of $y_2(0.2)$ by performing Euler's method with step $h = 0.1$.

Solution: Let $\tilde{y_2}$ be the approximation. We have 2 steps in Euler's method since the stepsize $h = 0.1$:

$$\tilde{y_2}(0.1) = y_2(0) + 0.1y_2'(0)$$
$$\tilde{y_2}(0.2) = \tilde{y_2}(0.1) + 0.1y_2'(0.1).$$

Plugging numbers in the first equation gives:

$$\tilde{y_2}(0.1) = 0 + 0.1e^0 = 0.1.$$

Then for the second equation we have:

$$\tilde{y_2}(0.2) = 0.1 + 0.1e^{0.1^2} = 0.1 + 0.1e^{0.01}.$$

4. (20 points) Consider the second-order differential equation:

$$y''(t) + 3y'(t) - 4y(t) = 16te^{3t}.$$

(a) (10 points) Find *one* solution to the differential equation.

Solution: Let us find the particular solution by the method of undetermined coefficients: since the right hand side is of the form te^{3t}, our guess will be:

$$y_p(t) = (At + B)e^{3t}.$$

Its 1st and 2nd derivatives are:

$$\begin{aligned} y_p'(t) &= 3Ate^{3t} + (3B + A)e^{3t} \\ y_p''(t) &= 9Ate^{3t} + (9B + 6A)e^{3t}. \end{aligned}$$

Plugging them in the differential equation gives linear equations for A, B:

$$\begin{aligned} 14A &= 16 \\ 14B + 9A &= 0, \end{aligned}$$

whose solution is $A = 8/7$ and $B = -36/49$. Then we have

$$y_p(t) = \frac{8}{7}te^{3t} - \frac{36}{49}e^{3t}.$$

(b) (10 points) Find *all* solutions to the differential equation above.

Solution: This is a second-order *linear* differential equation, and thus any solution will be a sum of a general solution to the homogeneous problem *with* a particular solution. We have already found a particular solution, which means all solutions can be obtained by solving the homogeneous equation

$$y''(t) + 3y'(t) - 4y(t) = 0.$$

The characteristic equation is:

$$\lambda^2 + 3\lambda - 4 = 0,$$

which has 2 real roots: $\lambda_1 = 1, \lambda_2 = -4$. So the general solution of the original equation is:

$$y(t) = y_p(t) + C_3 e^t + C_4 e^{-4t} = \frac{8}{7} t e^{3t} - \frac{36}{49} e^{3t} + C_3 e^t + C_4 e^{-4t},$$

where C_3, C_4 are arbitrary constants.

5. (20 points) For each of the five sentences below, circle the unique correct answer.

   (a) (2 points) The following is a solution of $y''(t) = -4y(t)$:

   (1) e^{4t} (2) e^{2t} (3) $\sin(4t)$ (4) $\sin(2t)$

   Solution: 4 is correct since $(\sin(2t))'' = -4\sin(2t)$.

   (b) (2 points) All constant solutions to the differential equation $y''(t) = y^3(t) - y(t)$ are:

   (1) $0, 1$ (2) 0 (3) There are no constant solutions. (4) $0, 1, -1$

   Solution: 4 is correct since $0, 1, -1$ are 3 roots of $y^3 - y = 0$.

   (c) (2 points) The system given by $y''(t) + 4y'(t) + 4y(t) = 0$ is:

   (1) Underdamped. (2) Critically Damped. (3) Overdamped.

   Solution: 2 is correct since the characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ has only one real root: $\lambda = -4$.

   (d) (2 points) The long-term behaviour of the solutions of $y''(t) + 4y'(t) + 3y(t) = 0$

   (1) Does never exist. (2) Exists for some solutions but not for all solutions.

   (3) Always exists but depends on the solution. (4) It is always zero.

   Solution: 4 is correct since the general solution is $y(t) = C_1 e^{-t} + C_2 e^{-3t}$.

   (e) (2 points) The global error in Euler's method with step h is proportional to:

   (1) h^{-1} (2) h (3) h^2 (4) $\ln(h)$

   Solution: 2 is correct, since the local error is of order h^2 and there are an order of h^{-1} steps in Euler's method, thus the global error is $h = h^2 \cdot h^{-1}$.