Abstract. These are the solutions to Problem Set 4 for MAT 108 in the Fall Quarter 2020. The problems were posted online on Wednesday Nov 4 and due Friday Nov 13.

Problem 1. (Proposition 8.53) Prove that every non-empty subset of \( \mathbb{R} \) that is bounded below has a greatest lower bound.

Solution. Let \( A \) be a nonempty subset of \( \mathbb{R} \) that is bounded below. Construct a new set \( \tilde{A} = \{-a | a \in A\} \). This set is bounded above because \( l \) being a lower bound of \( A \) implies \( -l \) is an upper bound of \( \tilde{A} \). In other words, \( l \leq a \) for all \( a \in A \) and negating this gives \( -l \geq -a \) for all \( a \in A \). By the Completeness Axiom, \( s = \sup \tilde{A} \) exists. We claim \( -s = \inf(A) \). By definition, \( s \) being a supremum of \( \tilde{A} \) implies \( -a \leq s \) for all \( a \in \tilde{A} \). Multiply this inequality by \(-1\) to get \( a \geq -s \). Hence, \( -s \) is a lower bound of \( A \). Moreover, it has to be our greatest lower bound. If not, then suppose \( -t \) is the infimum of \( A \) so \( -t \leq a \) for all \( a \in A \). This would imply \( t \geq -a \), i.e. the supremum of \( \tilde{A} \) is \( t \), a contradiction.

Problem 2. (20 points, 5 each) Find the least upper bound \( \sup(A) \), and the greatest lower bound \( \inf(A) \) of the following subsets of the real numbers \( \mathbb{R} \):

(a) \( A = (-3.2, 7) \subseteq \mathbb{R}, \) i.e. \( A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x < 7\} \subseteq \mathbb{R}. \)

(b) \( B = (-3.2, 7] \subseteq \mathbb{R}, \) i.e. \( A = \{x \in \mathbb{R} : -3.2 < x \text{ and } x \leq 7\} \subseteq \mathbb{R}. \)

(c) \( C = (0, \infty) \subseteq \mathbb{R}, \) i.e. \( A = \{x \in \mathbb{R} : 0 < x\} \subseteq \mathbb{R}. \)

(d) \( D = (-\infty, 4] \subseteq \mathbb{R}, \) i.e. \( A = \{x \in \mathbb{R} : x \leq 4\} \subseteq \mathbb{R}. \)

Solution. We will make use of the fact that the average of two distinct real numbers lies strictly between those two numbers. That is, for real numbers \( a < b \), we have

\[
a = \frac{a}{2} + \frac{a}{2} < \frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2} = b,
\]

so

\[
(0.1) \quad a < \frac{a+b}{2} < b.
\]

(a) We claim that \( \inf(A) = -3.2 \) and \( \sup(A) = 7 \). It is clear from the definition of \( A \) that these give a lower bound and upper bound, respectively. Let \( u \) be a lower bound for \( A \), and suppose for the sake of contradiction that \( u > -3.2. \)
Since $u$ is a lower bound for $A$, we also have
$$u \leq 0 < 7.$$  
Consider the average $r := \frac{-3.2 + u}{2}$, which, by (0.1) satisfies
$$-3.2 < r < u < 7,$$
so $r \in A$. Since $r < u$, this contradicts the fact that $u$ is a lower bound, so we conclude that $u \leq -3.2$ after all. Therefore, $-3.2$ is the greatest lower bound for $A$, as desired.

Similarly, Let $v$ be an upper bound for $A$, and suppose for the sake of contradiction that $v < 7$. Since $v$ is an upper bound for $A$, we also have
$$v \geq 0 > -3.2.$$  
Consider the average $r := \frac{v + 7}{2}$, which, by (0.1) satisfies
$$-3.2 < v < r < 7,$$
so $r \in A$. Since $r > v$, this contradicts the fact that $v$ is a lower bound, so we conclude that $v \geq -3.2$ after all. Therefore, $-3.2$ is the least upper bound for $A$, as desired.

(b) The proof is nearly identical to Part (a). We claim that $\inf(B) = -3.2$ and $\sup(B) = 7$. It is clear from the definition of $B$ that these give a lower bound and upper bound, respectively. Let $u$ be a lower bound for $B$, and suppose for the sake of contradiction that $u > -3.2$. Since $u$ is a lower bound for $B$, we also have
$$u \leq 7.$$  
Consider the average $r := \frac{-3.2 + u}{2}$, which, by (0.1) satisfies
$$-3.2 < r < u \leq 7,$$
so $r \in B$. Since $r < u$, this contradicts the fact that $u$ is a lower bound, so we conclude that $u \leq -3.2$ after all. Therefore, $-3.2$ is the greatest lower bound for $B$, as desired.

Similarly, Let $v$ be an upper bound for $B$, and suppose for the sake of contradiction that $v < 7$. Since $v$ is an upper bound for $B$, we also have
$$v \geq 0 > -3.2.$$  
Consider the average $r := \frac{v + 7}{2}$, which, by (0.1) satisfies
$$-3.2 < v < r < 7,$$
so $r \in B$. Since $r > v$, this contradicts the fact that $v$ is a lower bound, so we conclude that $v \geq -3.2$ after all. Therefore, $-3.2$ is the least upper bound for $B$, as desired.Alternatively, notice that $\max(B) = 7$, so a Proposition from Discussion 6 tells us that $\sup(B) = 7$.

(c) We claim that $\inf(C) = 0$ and that $C$ has no supremum. The proof the the former is by now standard. It is clear from the definition of $C$ that $0$ is a lower bound. Let $u$ be a lower bound for $C$, and suppose for the sake of contradiction that $u > 0$. Consider the average $r := \frac{0 + u}{2}$, which, by (0.1) satisfies
$$0 < r < u,$$
so \( r \in C \). Since \( r < u \), this contradicts the fact that \( u \) is a lower bound, so we conclude that \( u \leq 0 \) after all. Therefore, 0 is the greatest lower bound for \( C \), as desired.

To show that \( C \) has no supremum, we show that it has no upper bounds (this suffices because suprema are, in particular, upper bounds). Indeed, let \( x \in \mathbb{R} \). If \( x \leq 0 \), then \( x < 1 \), but \( 1 \in C \), so \( x \) is not an upper bound for \( C \). Otherwise, \( x > 0 \), and we have \( x < x + 1 \), but \( x + 1 > x \) is in \( C \), so \( x \) is again not an upper bound. Having excluded all possible real numbers as upper bounds, we conclude that \( C \) has no upper bound.

(d) We claim that \( \text{sup}(D) = 4 \) and that \( D \) has no infimum. The proof the the former is by now standard. It is clear from the definition of \( D \) that 4 is an upper bound. Let \( v \) be an upper bound for \( D \), and suppose for the sake of contradiction that \( v < 4 \). Consider the average \( r := \frac{v + 4}{2} \), which, by (0.1) satisfies

\[
v < r < 4,
\]

so \( r \in D \). Since \( r > v \), this contradicts the fact that \( v \) is a lower bound, so we conclude that \( v \geq 4 \) after all. Therefore, 4 is the greatest lower bound for \( C \), as desired. Alternatively, notice that \( \text{max}(D) = 4 \), so a Proposition from Discussion 6 tells us that \( \text{sup}(D) = 4 \).

To show that \( D \) has no infimum, we proceed as in Part (c) by showing that it has no lower bound (this suffices because infima are, in particular, lower bounds). Indeed, let \( x \in \mathbb{R} \). If \( x > 4 \), then—because \( 4 \in D \)—\( x \) is not a lower bound for \( D \). Otherwise, \( x \leq 4 \), and we have \( x > x - 1 \), but \( x - 1 < x \leq 4 \) is in \( D \), so \( x \) is again not a lower bound. Having excluded all possible real numbers as lower bounds, we conclude that \( D \) has no lower bound.

**Problem 3.** (10+10 points) Consider the set of real numbers

\[
N = \left\{ 3 - \frac{1}{n} : n \in \mathbb{N} \right\}.
\]

Find \( \text{inf}(N) \) and \( \text{sup}(N) \).

**Solution.** We claim \( \text{sup}(N) = 3 \) and \( \text{inf}(N) = 2 \). Since \( 3 > 3 - \frac{1}{n} \) for all \( n \in \mathbb{N} \), we know 3 is un upper bound for \( N \). We know for each \( \varepsilon > 0 \), there exists an \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon \). Then, \( 3 - \frac{1}{n} > 3 - \varepsilon \) so \( 3 - \varepsilon \) is not an upper bound for any \( \varepsilon > 0 \). Thus, 3 must be our least upper bound. Now we prove the infimum is 2. Note that 2 is a lower bound. Moreover, \( 3 - \frac{1}{n+1} > 3 - \frac{1}{n} \geq 2 \) because \( \frac{1}{n+1} < \frac{1}{n} \) for all \( n \in \mathbb{N} \). Therefore, 2 is our greatest lower bound.
Problem 4. Consider the two following subsets of the real numbers
\[ S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}, \quad T = \left\{ \frac{2n+1}{n+1} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}. \]

Show that \( \sup(S) = 1 \), \( \sup(T) = 2 \) and \( \inf(T) = \frac{3}{2} \). Find \( \inf(S) \).

Solution. Define
\[ s_n = \frac{n}{n+1} \quad \text{and} \quad t_n = \frac{2n+1}{n+1}. \]

Then we have the sequences \((s_n)_{n \in \mathbb{N}}\) and \((t_n)_{n \in \mathbb{N}}\). Note that
\[ t_n = \frac{2n+1}{n+1} = \frac{n+n+1}{n+1} = \frac{n}{n+1} + \frac{n+1}{n+1} = s_n + 1, \]
so our sets are \( S = \{s_n : n \in \mathbb{N}\} \) and \( T = \{s_n + 1 : n \in \mathbb{N}\} \). We show that the sequence \((s_n)_{n \in \mathbb{N}}\) is monotone. Indeed, for each \( n \in \mathbb{N} \), we have
\[
\begin{align*}
s_{n+1} - s_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\
&= \frac{(n+1)(n+1) - (n+2)n}{(n+2)(n+1)} \\
&= \frac{1}{(n+2)(n+1)} \\
&\geq 0,
\end{align*}
\]
so \( s_{n+1} \geq s_n \). In particular,
\[
\begin{align*}
s_n - s_1 &= \frac{n}{n+1} - \frac{1}{2} = \frac{2n - (n+1)}{2(n+1)} = \frac{n-1}{2(n+1)} \geq 0,
\end{align*}
\]
since \( n \geq 1 \), so \( \frac{1}{2} = s_1 \leq s_n \). Therefore, \( \frac{1}{2} \in S \) is a lower bound for \( S \), and hence \( \inf(S) = \frac{1}{2} \) by a Proposition from Discussion 6.

By the proof of the Monotone Convergence Theorem, the limit of \((s_n)_{n \in \mathbb{N}}\) exists and is equal to \( \sup(S) \), so we now prove that \( \lim_{n \to \infty} s_n = 1 \). Let \( \varepsilon > 0 \), and let \( n_0 \in \mathbb{N} \) be such that \( \frac{1}{n_0} < \varepsilon \). Then, for all \( n \geq n_0 \) we have
\[
\begin{align*}
|1 - s_n| &= \left| 1 - \frac{n}{n+1} \right| = 1 - \frac{n}{n+1} = \frac{(n+1) - n}{n+1} = \frac{1}{n+1} \leq \frac{1}{n_0} < \varepsilon.
\end{align*}
\]

Note that in the second equality above we used the fact that \( n < n+1 \), which rearranges to \( 1 - \frac{n}{n+1} > 0 \). This completes the proof that \( \sup(S) = \lim_{n \to \infty} s_n = 1 \).

The calculations for \( T \) follow from those for \( S \). The sequence \((t_n)_{n \in \mathbb{N}}\) is monotone because
\[
\begin{align*}
t_{n+1} - t_n &= (s_{n+1} + 1) - (s_n + 1) = s_{n+1} - s_n \geq 0
\end{align*}
\]
for all \( n \in \mathbb{N} \). In particular,
\[
\begin{align*}
t_n - t_1 &= (s_n + 1) - (s_1 + 1) = s_n - s_1 \geq 0
\end{align*}
\]
so \( \frac{3}{2} = \frac{3}{2} \in T \) is a lower bound for \( T \), and hence \( \inf(T) = \frac{3}{2} \) by a Proposition from Discussion 6.
By the proof of the Monotone Convergence Theorem, the limit of \((t_n)_{n \in \mathbb{N}}\) exists and is equal to \(\sup(T)\), so we now prove that \(\lim_{n \to \infty} t_n = 2\). Let \(\varepsilon > 0\), and let \(n_0 \in \mathbb{N}\) be such that \(\frac{1}{n_0} < \varepsilon\). Then, for all \(n \geq n_0\) we have

\[|2 - t_n| = |2 - (s_n + 1)| = |1 - s_n| < \varepsilon.\]

This completes the proof that \(\sup(T) = \lim_{n \to \infty} t_n = 2\).

**Problem 5.** (10+5+5 points) Find an upper bound for each of the following three sets:

\[X = \left\{ \left(1 + \frac{1}{n}\right)^n : n \in \mathbb{N} \right\}, \quad Y = \left\{ \left(1 + \frac{1}{n^2}\right)^n : n \in \mathbb{N} \right\}, \quad Z = \left\{ \left(1 + \frac{1}{n}\right)^{n^2} : n \in \mathbb{N} \right\}.\]

*Hint:* Consider the following expansion

\[
\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right).
\]

**Solution.**

(i) Let’s look at the expansion:

\[
\sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \left(1 - \frac{3}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right).
\]

In discussion, we proved that as \(n\) becomes larger, the value of \(\frac{1}{n}\) becomes smaller and the infimum of the set \(\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}\) is thus 0. Therefore, each term in the parenthesis is bounded above by 1 so it suffices to consider

\[
\sum_{k=0}^{n} \frac{1}{k!}.
\]

Therefore, we have the following.

\[
\left(1 + \frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2n} + \cdots
\]

\[
= 1 + \sum_{k=0}^{\infty} \frac{1}{2^k}
\]

\[
= 3
\]

The last equality follows since the sum of the infinite geometric series \(\sum_{k=0}^{\infty} \frac{1}{2^k}\) is \(\frac{1}{1-\frac{1}{2}} = 2\).
(ii) Notice that 
\[
\left(1 + \frac{1}{n^2}\right)^n = \left(\left(1 + \frac{1}{n^2}\right)^{n^2}\right)^{1/n}.
\]

We know \(\left(1 + \frac{1}{n^2}\right)^{n^2}\) is bounded above by 3 from part (i). (If it’s difficult to see, replace \(n^2\) with a new variable \(z\), for instance.) It is enough to then consider \(3^{1/n}\). Using what we know about the behavior of \(\frac{1}{n}\), we conclude it is bounded above by 3.

(iii) We claim that this set has no upper bound. Notice that 
\[
c_n := \left(1 + \frac{1}{n}\right)^{n^2} = \left(\left(1 + \frac{1}{n}\right)^{n^2}\right)^n.
\]

This is similar to part (ii). By the Binomial Theorem, we have 
\[
\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \binom{n}{0} \frac{1}{n^0} + \binom{n}{1} \frac{1}{n^1} + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} = 1 \cdot 1 + n \cdot \frac{1}{n} + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} \geq 2,
\]
so \(c_n \geq 2^n\). It now suffices to show that the sequence \((2^n)_{n \in \mathbb{N}}\) is unbounded, which we prove by showing that \(2^n \geq n\) using induction. (This proves it is not bounded above since the natural numbers is not bounded above.) For the base case, we have \(2^1 \geq 1\), which is true. Now assume \(2^k \geq k\). We then have 
\[
2^{k+1} = 2^k \cdot 2 > k \cdot 2 \geq k + 1.
\]
The last inequality follows because \(2k \geq k + 1\) can be rewritten as \(k \geq 1\), which is true.

**Problem 6.** (10+10 points) Consider the subset \(C_0 = [0, 1] \subseteq \mathbb{R}\). Recursively, define the sets 
\[
C_{n+1} = \frac{C_n}{3} \cup \left(\frac{2}{3} + \frac{C_n}{3}\right),
\]
for \(n \geq 1\), where, if we let \(A = [a, b]\), then the notation \(A/3\) describes the interval \([a/3, b/3]\) and the notation \(A + 2/3\) describe the interval \([a + 2/3, b + 2/3]\).

(a) Describe and draw the sets \(C_1, C_2, C_3\) and \(C_4\) as a union of explicit intervals.

(b) Show that the intersection \(\cap_{n=1}^{\infty} C_n\) is non-empty.

**Solution.** Here is the extension of the notations \(A/3\) and \(A + 2/3\) for arbitrary sets. Let \(X \subseteq \mathbb{R}\) be an arbitrary subset, and let \(c\) be any real number. Then we define the new sets 
\[
c \cdot X := \{c \cdot x : x \in X\} \subseteq \mathbb{R} \quad \text{and} \quad X + c := \{x + c : x \in X\} \subseteq \mathbb{R}.
\]
For $c \neq 0$, we also define $\frac{X}{c} := \frac{1}{c} \cdot X$.

(a) The set $C_{n+1}$ is obtained from $C_n$ by scaling all of $C_n$ down to fit inside $[0, \frac{1}{3}]$, and then repeating this scaled copy in the translation to $[\frac{2}{3}, 1]$. It follows that $C_{n+1}$ is given by deleting the open middle third of each interval in $C_n$. Explicitly,

\[
C_0 = [0,1] \\
C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\
C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{9}] \cup [\frac{7}{9}, 1] \\
C_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{5}{9}] \cup [\frac{5}{9}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{8}{9}] \\
C_4 = [0, \frac{1}{81}] \cup [\frac{2}{81}, \frac{1}{27}] \cup [\frac{2}{27}, \frac{7}{81}] \cup [\frac{8}{81}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{19}{81}] \cup [\frac{20}{81}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{81}] \cup [\frac{26}{81}, \frac{1}{3}] \\
\quad \cup [\frac{2}{3}, \frac{50}{81}] \cup [\frac{52}{81}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{61}{81}] \cup [\frac{62}{81}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{73}{81}] \cup [\frac{74}{81}, \frac{25}{27}] \cup [\frac{26}{27}, \frac{79}{81}] \cup [\frac{80}{81}, 1].
\]

These are illustrated in Figure 1 below, taken from georgcantorbyelithompson.blogspot.com

![Figure 1](image-url)

**Figure 1.** The sets $C_0, C_1, C_2, C_3,$ and $C_4$.

(b) We will show that $0 \in C_n$ for all integers $n \geq 0$ by induction on $n$. For our base case $n = 0$, we have $0 \in [0,1] = C_0$ (it’s important that we’re working with closed intervals). As our inductive hypothesis, suppose $0 \in C_n$ for some integer $n \geq 0$. Then

\[
0 = \frac{0}{3} \in \frac{C_n}{3} \subseteq C_{n+1},
\]

so $0 \in C_{n+1}$. We conclude that $0 \in C_n$ for all $n \geq 0$, so $0 \in \bigcap_{n=0}^{\infty} C_n$, and consequently $\bigcap_{n=0}^{\infty} C_n$ is not empty.

**Note:** The set $C_n \subseteq \mathbb{R}$ is a union of $2^n$ disjoint closed intervals. The above argument works similarly to show that any of the endpoints of these intervals persist in the further sets $C_{n+1}, C_{n+2},$ etc. (and of course, they’re contained in $C_{n-1}, C_{n-2},$ etc. as well, since $C_0 \supset C_1 \supset C_2 \cdots$).

So each of these $2 \cdot 2^n$ points in the set $C_n$ is in the intersection $\bigcap_{n=0}^{\infty} C_n$, and consequently the set $C := \bigcap_{n=0}^{\infty} C_n$ has infinitely many points! In fact, these persisting endpoints are the *only* elements of $C$. Notice the $2^{n+1}$ endpoints from $C_n$ can all be written as rational numbers with common denominator $3^n$.

The set $C := \bigcap_{n=0}^{\infty} C_n$ is called the Cantor set, and it exhibits a wide variety of strange phenomena that can occur in the real numbers $\mathbb{R}$.