Lecture 10: Equivalence Relations

Definition: Let \( X \) be a set \( (X \neq \emptyset) \), an equivalence relation on \( X \) is a relation on \( X \) such that:
1. Reflexive: \( a \sim a \) for all \( a \in X \).
2. Symmetric: If \( a \sim b \), then \( b \sim a \).
3. Transitive: If \( a \sim b \) and \( b \sim c \), then \( a \sim c \).

Example: \( X = \mathbb{Z} \), the relation \( a \sim b \) is even, \( a - b \) is divisible by 2.

Non-example: \( X = \mathbb{Z} \), relation \( a \sim b \) if \( a < b \). This is not an equivalence relation.

Proposition 6.24: (module n) Let \( \mathbb{Z} \) be the set of integers. Then, \( \mathbb{Z} \) is an equivalence relation if \( a \sim b \) is divisible by \( n \). Then:
1. It is reflexive: \( a \sim a \) is divisible by \( n \).
2. It is symmetric: If \( a \sim b \), then \( b \sim a \).
3. It is transitive: If \( a \sim b \) and \( b \sim c \), then \( a \sim c \).

\[ a \equiv b \pmod{n} \iff a - b = kn \] for each \( k \in \mathbb{Z} \).

Example: \( x \equiv 3 \pmod{5} \), \( 0 \equiv 4 \), \( 11 \equiv 2 \), \( 19 \equiv -1 \), \( 3 \equiv 4 \).