Lecture 14: The Real Numbers $\mathbb{R}$ (Chapter VIII)

We will consider a new set, denoted $\mathbb{R}$, with operations $(+, \cdot)$ satisfying Axioms 8.1-8.4 (in tutorial: please 2). The new axiom for product:

**Axiom 8.5:** $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$, s.t. $x \cdot y = 1$.

$y$ is denoted $x^{-1}$, or $\frac{1}{x}$, informally "we can use fractions".

Not true for $\mathbb{Z}$: if $3 \cdot \frac{1}{3} = 1$, then $\frac{1}{3} \in \mathbb{Z}$.

Similarly to how we introduced $\mathbb{Q} \subseteq \mathbb{R}$, $\exists \mathbb{R}_{\geq}$ a subset, called "positive real numbers".

**Axioms for $\mathbb{R}_{\geq}$:**
1. $x, y \in \mathbb{R}_{\geq}$, then $x + y \in \mathbb{R}_{\geq}$.
2. $x \cdot y \in \mathbb{R}_{\geq}$.
3. $x > 0$ or $y > 0$.
4. $(\exists) x \in \mathbb{R}_{\geq}, 0 < x < 1$ (only 1 happy).

Recall that $\mathbb{N} \subseteq \mathbb{Z}$ had a smallest element.

**Then:** the set $\mathbb{R}_{\geq}$ has no smallest element.

**Proof:** by contradiction, suppose $s \in \mathbb{R}_{\geq}$ is the smallest element.

Consider the real number $\frac{s}{2} \in \mathbb{R}_{\geq}$.

in fact: $\frac{s}{2} \in \mathbb{R}_{\geq}$ because $2 \in \mathbb{R}_{\geq}$, so $\frac{1}{2} \in \mathbb{R}_{\geq}$ and thus $\frac{s}{2} \in \mathbb{R}_{\geq}$.

Thus, $\frac{s}{2} < s$.

so $s \in \mathbb{R}_{\geq}$ is not the smallest element.

§2. Bounds for subsets

as crucial to talk about limits and other ideas.

a subset $A \subseteq \mathbb{R}$ is said to be bounded above by $b \in \mathbb{R}$ if $\forall a \in A$, $a \leq b$.

A subset $A \subseteq \mathbb{R}$ is said to be bounded below by $c \in \mathbb{R}$ if $\forall a \in A$, $a \geq c$.

As a set, $A$ has many upper (and lower) bounds:

**Def.** let $A \subseteq \mathbb{R}$ be a subset. The least upper bound is called the supremum.

The greatest lower bound is called the infimum.

leads to the "completeness" axiom!