Lecture 18: Recursive Limits & Square Roots

Suppose \((x_n)_{n \in \mathbb{N}}\) is a sequence of real numbers defined recursively: \(x_1 = \frac{1}{2} (x_0 + \frac{1}{x_0})\) with \(x_0 > 2\) and \(x_n = \frac{1}{2} (x_{n-1} + \frac{1}{x_{n-1}})\) for \(n \geq 2\).

Question: How to show \((x_n)\) converges? If so, what is \(\lim_{n \to \infty} x_n\)?

Uniqueness of limits: \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n\)

Today's running example: \(x_{n+1} = \frac{1}{2} (x_n + \frac{1}{x_n}), \ x_0 > 2\)

Examine: Study the recursive sequence \(x_{n+1} = x_n (2 - x_n), \ x_0 = \sqrt{2}\).

Consider the set \(X = \{x \in \mathbb{R} : x^2 < 2\}\). First, \(X\) is bounded above \(\Rightarrow \sup X \in \mathbb{R}\).

Definition: We define \(\sqrt{2} = \sup \{x \in \mathbb{R} : x^2 < 2\}\).

Theorem 10.25: \(X\) is bounded above (say \(M\)) \(\Rightarrow \sup \{x \in \mathbb{R} : x^2 < 2\} = \sqrt{2}\).

Also, \(\sup X\) is such that:

1. \(\sup X > 0\)
2. \(\sup X = 2\)

Check proof in the textbook.

Q.2 Square roots: how do we define \(\sqrt{2}\)? (any \(r^2\) gives the same \(r > 0, \forall r \in \mathbb{R}\))

First, show \((x_n)\) convergent. We use Monotone Convergence Theorem, which we can do if \(x_n\) is bounded below and increasing.

(i) Bounded below: since \(x_0 > 2\) and sum of positive numbers, \(x_n > 0\).

(ii) Increasing: by induction, \(x_{n+1} > x_n\).

Substituting in (1): \(L = \frac{1}{2} (L + \frac{1}{L}) \Rightarrow 2L^2 - L - 2 = 0\) \(\Rightarrow L = \sqrt{2}\) (true).

There are properties which we know and use about \(\sqrt{2}\), \(\sqrt{2} > 0\) and \(\sqrt{2}\) is irrational for some real numbers.