Lecture 5 : Problems & Applications

Problems & Applications

1. Today, we finish Induction (recursion was next!)

- Sum of first n odd numbers
  \[ \sum_{i=1}^{n} (2i-1) = n^2 \]
  (by induction)

2. Geometric Progressions
   - See real-world examples
   - How many regions if we tile n x n grid?

Problem 1: Show that \( \sum_{k=1}^{n} (2i-1) = n^2 \), \( \forall n \in \mathbb{N} \).

\[ \text{Sol: } \]
- **Base Case**: \( n = 1 \), and the formula reads \( \sum_{i=1}^{1} (2i-1) = 1^2 \), so \( n=1 \) which is true.

- **Induction Step**: Assume \( \sum_{i=1}^{n} (2i-1) = n^2 \), we want to show \( \sum_{i=1}^{n+1} (2i-1) = (n+1)^2 \).

\[ \sum_{i=1}^{n+1} (2i-1) = \sum_{i=1}^{n} (2i-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2 \]

Problem 2: Show that \( 2^n \leq n! \) is true for \( \forall n \in \mathbb{N} \), \( n \geq 4 \).

\[ \text{Sol: } \]
- By induction, we need to verify 2 steps:
  - **Base Case**: \( n=4 \), need to check \( 2^4 \leq 4! \). Since \( 16 \leq 24 \), the base case is true.
  - **Induction Step**: Assume \( 2^k \leq k! \), we want to show \( 2^{k+1} \leq (k+1)! \).

\[ \sum_{i=1}^{n} (2i-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2 \]

Problem 3 (Euclid): There are infinitely many primes.

\[ \text{Sol: } \]
- By contradiction, assume the opposite: \( \exists \) finitely many primes. 
- Try to reach a contradiction.

If we have finitely many primes, then we can write them as \( p_1, p_2, \ldots, p_k \).

Now consider a number \( \prod_{i=1}^{k} p_i + 1 \). Since it is not divisible by any of the primes up to \( p_k \), it must be a prime. 

But if \( p \mid \prod_{i=1}^{k} p_i + 1 \), then \( p \\
\frac{\prod_{i=1}^{k} p_i + 1}{p} \)