Lecture 8: Homogeneous Linear Recursions

**Def:** Let \((x_n)\) be a sequence, then \((x_n)\) is defined by a linear recursion if \(x_n = \sum_{i=1}^{k} a_i x_{n-i} + c, \quad a_i, c \in \mathbb{Z}\)

- **Ex.** 1: \(x_n = x_{n-1} + x_{n-2}\) with \(x_1 = 1, x_2 = 2\).
  - Then \(x_n = 2x_{n-1} - x_{n-2}\).
  - **Ex.** 2: \(x_n = x_{n-1} + x_{n-2}\) with \(x_1 = 1, x_2 = 1\).
    - Then \(x_n = 2x_{n-1} - x_{n-2}\).
  - **Ex.** 3: \(x_n = x_{n-1} + x_{n-2}\) with \(x_1 = 1, x_2 = 2\).
    - Then \(x_n = 2x_{n-1} - x_{n-2}\).

**§ 1. Goal:** Finding a closed formula

**Ex.:** \((x_n) = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)\) Fibonacci, \(x_n = x_{n-1} + x_{n-2}\) with \(x_1 = 1, x_2 = 1\).

Given the terms up to \(x_n\), finding \(x_n\) is not easy. The challenge is to find a general formula for \(x_n\), i.e., \(x_n = \text{formulæ only in terms of } n\).

Today, we will study a general formula for recurrences of the form:

\[
(\#) \quad x_n = A \cdot x_{n-1} + B \cdot x_{n-2}, \quad A, B \in \mathbb{Z}.
\]

We use a new technique called “characteristic polynomials.”

**§ 2. Characteristic Polynomial:** Given the recurrence \((\#)\) \(x_n = A \cdot x_{n-1} + B \cdot x_{n-2}\),

- **Def.** The char. poly. of \((\#)\) is \(p(r) = r^2 - Ar - B\).
- Its roots are \(r_1, r_2\), and we assume \(r_1 \neq r_2\).

Then:
- \(x_n = C \cdot r_1^n + D \cdot r_2^n\), with \(C, D \in \mathbb{R}\).
- Or \(\frac{f_{n+2}}{f_n} \rightarrow \phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad (1 - \sqrt{5})\).

**Ex:** How to compute \(C, D \in \mathbb{Z}\)?

- Given \(p(r) = r^2 - Ar - B\) and its roots \(r_1, r_2\), i.e., \(r_1, r_2\).
- Then, \(x_n = C \cdot r_1^n + D \cdot r_2^n\), with \(C, D \in \mathbb{R}\) determined by the values \(x_1, x_2\).

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**§ 3. The Fibonacci Sequence & Binet’s Formula:** Let \(f_n = f_{n-1} + f_{n-2}\) with \(f_1 = f_0 = 1\).

Then \((f_n) = (1, 2, 3, 5, 8, 13, 21, \ldots)\). Can we find formulas for \(f_n\)?

**Sol.** By solving, we need the roots \(r_1, r_2\) of the characteristic polynomial \(p(r) = r^2 - Ar - B\).

Then, \(r_1, r_2\), i.e., \(r_1, r_2\).

Then, \(x_n = C \cdot r_1^n + D \cdot r_2^n\), with \(C, D \in \mathbb{R}\) determined by the values \(x_1, x_2\).

**Ex:** How to compute \(C, D \in \mathbb{Z}\)?

- Given \(p(r) = r^2 - Ar - B\) and its roots \(r_1, r_2\), i.e., \(r_1, r_2\).
- Then, \(x_n = C \cdot r_1^n + D \cdot r_2^n\), with \(C, D \in \mathbb{R}\) determined by the values \(x_1, x_2\).