This examination document contains 5 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, the Internet, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (25 points) Show that the following inequalities hold:

(a) (15 points) Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}, \quad \forall n \in \mathbb{N}.$$ 

(b) (10 points) For $n \geq 6$ and $n \in \mathbb{N}$, show that

$$5n + 5 \leq n^2.$$
2. (25 points) Solve the following two parts:

(a) (10 points) Consider the sequence \( (x_n)_n \in \mathbb{N} \) given by the recursion

\[
x_{n+1} = x_n + (n - 1), \quad x_1 = 19.
\]

Find \( x_{2020} \).

(b) (15 points) Consider the sequence \( (x_n)_n \), \( n \in \mathbb{N} \cup \{0\} \) defined recursively as

\[
x_n = 7x_{n-1} - 10x_{n-2}, \quad x_0 = 2, x_1 = 3.
\]

Find a closed formula for \( x_n \).
3. (25 points) Solve the following two parts:

   (a) (10 points) Show that the coefficient in front of $x^4y^{19}$ in $(x + y)^{23}$ is 8855.

   (b) (15 points) Consider the expression $(x + y)^n$, show that the coefficient in front of $x^k y^{n-k}$ is the same as the coefficient in front of $x^{n-k} y^k$. 
4. (25 points) Solve the following two problems:

(a) (15 points) Show that there exists no integers $x, y \in \mathbb{Z}$ such that

$$4x^3 - 7y^3 = 2003.$$ 

(b) (10 points) Show that the last two digits of $62^{48}$ are 96.