Problem 2.

(c) \( \lim_{n \to \infty} \frac{n^3}{n^2 + n} = \lim_{n \to \infty} \frac{n^3}{n(n + 1)} = \lim_{n \to \infty} \frac{n^2}{n + 1} = \lim_{n \to \infty} n = \infty \)

Set: \( \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \left| \frac{n!}{n^n} - 0 \right| < \varepsilon \text{ if } n > n_0 \)

i.e. show that \( \forall \varepsilon > 0 \) we have

\[
\frac{n!}{n^n} < \varepsilon \text{ for } n \text{ large enough.}
\]

The key is bounding above \( \frac{n!}{n^n} \) by something we know is small (it'll be \( \frac{1}{n} \))

\[
\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \frac{2}{n} \frac{1}{n} < \frac{1}{n}. \]

By Prop. 10.4, \( \exists n_0 \in \mathbb{N} \text{ s.t. } \frac{1}{n} < \varepsilon \text{ for } n > n_0 \). Hence, for the same \( n_0 \)

we have \( \left| \frac{n!}{n^n} - 0 \right| < \frac{1}{n} < \varepsilon \text{ for } n > n_0 \).

(Also, if you want, Prop. 10.4 \( \exists n_0 \text{ s.t. } \frac{1}{n} < \varepsilon \)

then \( n > n_0 \text{ satisfies } \frac{1}{n} < \frac{n}{n_0} < \varepsilon \).)

\[
(d) \lim_{n \to \infty} \frac{3n^2 - 1}{n^2 + n} = 3.
\]

Solve: we need that \( \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \left| \frac{3n^2 - 1}{n^2 + n} - 3 \right| < \varepsilon \text{ if } n > n_0 \).

How to get that?

\[
\left| \frac{3n^2 - 1}{n^2 + n} - 3 \right| < \frac{3}{n} \iff \left| \frac{3n^2 - 1 - 3n^2 - 3n}{n^2 + n} \right| < \frac{3}{n}
\]
\[
\frac{n^2 - 1}{n^2 + n} < \frac{1}{n} < \frac{n^2 + n}{n^2 - 1}
\]

\[\iff \quad \frac{3n+1}{n^2+n} < \frac{3}{n} \iff \frac{3n+3}{n^2+n} = \frac{3}{n}\]

By Prop. 10.4, \(\forall \varepsilon > 0\), \(\exists n_0 \in \mathbb{N} \text{ s.t. } \frac{1}{n_0} < \varepsilon\). Given our initial \(\varepsilon > 0\), apply Prop. 10.4 to \(\varepsilon_0 = \frac{\varepsilon}{3}\); then we get \(n_0 \in \mathbb{N} \text{ s.t. } \frac{1}{n_0} < \varepsilon_0 = \frac{\varepsilon}{3}\).

That means \(\frac{3}{n_0} \leq \varepsilon\).

By \(\implies \) \(\frac{3n+1}{n^2+n} - 3 < \frac{3}{n}\), we get \(\frac{3n+1}{n^2+n} - 3 < \frac{3}{n} < \varepsilon\) if \(n \geq n_0\).

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**Prob 3. (b):** 
\(y_n = \sum_{k=1}^{n} \frac{1}{k^2}\), increasing \(\checkmark\), bounded above?

\(y_n\) is \(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}\).

We know: \(\sum_{k=0}^{\infty} r^k < \infty\) if \(r < 1\). Geometric Series!
\[
\sum_{k=0}^{n} \frac{1}{2^k} = 2^n
\]

\[
\frac{1}{1-\frac{1}{2}}, \quad n \geq 1
\]

\[
y_n = \sum_{k=1}^{n} \frac{1}{k^2}, \quad \text{for each } R \text{ we have } \exists m \text{ s.t. }
2^m \leq k \leq 2^{m+1}, \text{ then } \frac{1}{k} \leq 2^{-m}
\]

**Prob. 5. (a)** Suppose \((x_n)\) is convergent, then \(\exists C \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0\)

we have
\[
|X_n - L| < \varepsilon \quad \text{for } n \gg 1.
\]

This is true \(\forall \varepsilon > 0\), it must be true for \(\varepsilon = 1\).

Then we have
\[
|X_n - L| \leq 1 \quad \text{for } n \geq n_0.
\]

\[
\iff |x_n| \leq |L| + 1 \quad \text{for } n \geq n_0
\]

Choose \(M := \max \{ |x_1, x_2, \ldots, x_{n_0}, |L| + 1 \}\), then this

bonds \(X_n, \forall n \in N\), i.e., taken care of \(n \geq n_0\)

\[
|X_n| \leq M.
\]

So \((x_n)\) is bounded. \(\blacksquare\)

**Prob. 5. (c)** \(\exists \text{ seq. } (x_n) \text{ bounded but no convergent.}\)

We give an example that this indeed possible.

\[
x_n = (-1)^n
\]

Then
\[
|X_n| = |(-1)^n| = 1 \leq 2
\]

\[2 \text{ is upper bounded}\]
Then $|x_n| = (-1)^n = 1 \leq 2$

Now need to show $x_n$ is not convergent.

9. (b) $x_n = (-1)^n$ is convergent. $\Rightarrow$ true and proven by $\varepsilon$-def, $\lim_{n \to \infty} x_n = 0$

(c) $(x_n)$ is s.t. $|x_n|$ converges, then $x_n$ converges? $\Rightarrow$ use Prop. 10.4!

$x_n = (-1)^n$ does not converge, but $y_n = |x_n| = 1$ so convergent: $\Rightarrow$ counterexample $3, 0, -3, 0, -3, \ldots$

(d) $(x_n), (y_n)$ unbounded, then $(z_n) = (x_n, y_n)$ unbounded.

Counterexample: $x_n = (0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots)$

$x_{3k+1} = 0$ and $x_{3k} = k$. This is **UNCOUNTABLE**.

$y_n = (1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, \ldots)$

$y_{3k+1} = k+1$, $y_{3k} = 0$. This is also unbounded.

$(z_n) = (x_n, y_n)$ is the sequence $z_n = 0 \forall n \in \mathbb{N}$.

In particular, $z_n$ bounded!