Prob. 3 of Proc Prob II

Last digit of \(4^{100}\)?

**Method 1:** We want \(4^{100} \equiv x \mod 10\), with \(0 \leq x \leq 9\).

\[
\begin{align*}
4^0 & \equiv 1 \\
4^1 & \equiv 4 \\
4^2 & \equiv 6 \\
4^3 & \equiv 4 \\
4^4 & \equiv 6 \\
4^5 & \equiv 4 \\
4^6 & \equiv 6 \\
4^7 & \equiv 4 \\
4^8 & \equiv 6 \\
\end{align*}
\]

- Powers of 4 have last digit 4 if exponent is odd
- And have last digit 6 if exponent is even

\[
\begin{align*}
4^{100} \equiv 6 \mod 10
\end{align*}
\]

**Method 2:** We want \(4^{100} \equiv x \mod 2\) AND \(4^{100} \equiv y \mod 5\) \(\Rightarrow\) wall tell us value \(\mod 10\)

1. \(4^{100} \equiv 0 \mod 2\)
2. \(4^{100} \equiv 1 \mod 5\) \(\leftrightarrow\) \(a \equiv a \mod p\), in addition if \(a \neq 0 \mod p\)
   - \((-1)^{100} \equiv 1 \mod 5\)
   - \(25 \equiv 1 \equiv 1 \mod 5\)

\[
\begin{align*}
4^{100} & \equiv 6 \mod 10
\end{align*}
\]

**Prob. 2 in Proc Prob II:**

(a) Show \(3 \not| 4^{100}\).

See: Opt 1 is directly. \(4^{100} \equiv 2^{100} \mod 3\), only 2's in prime dec, so done.

Opt 2: Work modulo 3 and show \(4^{100} \equiv 0 \mod 3\).

Indeed, \(4^{100} \equiv 1^{100} \equiv 1 \not\equiv 0 \mod 3\), so done.
Prob 5: Prove that there are infinitely many primes of the form $6k + 5$.

**Sol:** By contradiction, assume there are only finitely many primes of the form $6k + 5$. Call these $p_1, p_2, ..., p_n$. Now consider the number

$$S := 6(p_1 p_2 ... p_n) - 1,$$

which is of the form $6k + 5$ by construction.

Now, $S 
eq p_i$ because $S > p_i$. So $S$ is not a prime of the form $6k + 5$.

Since $S$ is divisible by primes, we will reach a contradiction if

$$\begin{align*}
(1) & \ p_i | S, \ 1 \leq i \leq N: \text{why? if } p_i | S \text{ then } p_i | 1 - \text{ violates } 6_k + 5 \rightarrow \\
(2) & \ S \text{ must be divided by } p_i: \text{ why? } S \text{ has a prime dec., prime must be } \frac{6}{p_i + 1} \text{ or } 6k - 1. \\
\end{align*}$$

If all divisors of $S$ were $6k+1$, then $S$ would be $6k+1$, but $S$ is $6k-1$, so $S$ a prime of the form $6k-1$ dividing it.

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**Prob 1 of 9.5 + 3**

$$a_n = \# \text{ ways to pair up } 2n \text{ players.}$$

Ex:

- $a_1 = 1$:
- $a_2 = 3 \cdot a_1$:
- $a_3 = 5 \cdot a_2$:
- $a_4 = 9 \cdot a_3$:
- $a_5 = 17 \cdot a_4$:
- $a_6 = 31 \cdot a_5$:
- $a_7 = 61 \cdot a_6$:
- $a_8 = 125 \cdot a_7$:
- $a_9 = 241 \cdot a_8$:

$$a_n = (2n-1) \cdot a_{n-1}.$$
Prob 4(b) Show \(5 \mid M^n - 6\), then \(M \in \mathbb{N}\)

Solve: We want to show \(M^n - 6 \equiv 0 \pmod{5}\).

Let's see \(M \equiv 1 \pmod{5}\), also \(6 \equiv 1 \pmod{5}\),

so \(M^n - 6 \equiv 1^n - 1 \equiv 0 \pmod{5}\ \square\)