MAT 108: PRACTICE PROBLEMS II

DEPARTMENT OF MATHEMATICS - UC DAVIS

Abstract. This document contains additional practice problems for the previous two weeks before the Oct 30 Midterm in the MAT-108 course during Fall 2020.

Purpose: The goal of this document is to provide practice problems on the topic of set theory and modular arithmetic covered in the lectures from Monday Oct 19 to Monday Oct 26. I have posted this document in order to help you practice problems on these topics, with a view towards the first Midterm Exam on Friday Oct 30. This document includes material on the following topics:

(i) Divisibility statements, and last digit computations.

(ii) Reduction modulo n of Equations over the Integers.

Note that the Midterm includes additional topics, covered in the first four weeks of the course. Thus in addition to the types of problems below, you should practice problems from the first set of Practice Problems.

Textbook: We are using "The Art of Proof: Basic Training for Deeper Mathematics" by M. Beck and R. Geoghegan.

Problem 1. Prove, using modular arithmetic, the following divisibility statements. Be clear which n are you choosing when you perform computations modulo n.

(a) $3|k^3 - k$, for all $k \in \mathbb{N}$,

(b) $11|k^{11} + 10k$, for all $k \in \mathbb{N}$,

(c) $23|k^{23} - k$, for all $k \in \mathbb{N}$,

(d) $6|k^3 - k$, for all $k \in \mathbb{N}$,

(e) $15|4^{2n+1} - 7^{4n-2}$, for all $n \in \mathbb{N}$,

Solution: Parts (a), (b) and (c) follow from Fermat's Little theorem modulo p, where p is correspondingly taken to be $3, 11$ and 23. In general,

$$p|k^p - k,$$

since $k^p \equiv k \mod p$ by Fermat's Little Theorem. In Part (a) we have

$$k^3 - k \equiv k - k \equiv 0 \mod 3.$$

In Part (b) we are reducing modulo 11 and we get

$$k^{11} + 10k \equiv k + 10k \equiv 11k \equiv 0 \mod p.$$

For Part (c) we reduce modulo 23 and we obtain

$$k^{23} - k \equiv k - k \equiv 0 \mod 23.$$

In order to work out divisibility by a composite number which is a product of distinct primes, such as $6 = 2\cdot 3$ and $15 = 3\cdot 5$, it suffices to show that each of the prime factors divides.

For Part (d), we need to work modulo 2 and modulo 3:

$$k^3 - k \equiv k^2 \cdot k - k \equiv k - k \equiv 0 \mod 2,$$
$$k^3 - k \equiv k - k \equiv 0 \mod 3.$$

For Part (e), we work modulo 3 and modulo 5:

$$4^{2n+1} - 7^{4n-2} \equiv (1)^{2n+1} - (1)^{4n-2} \equiv 1 - 1 \equiv 0 \mod 3,$$

$$4^{2n+1} - 7^{4n-2} \equiv (-1)^{2n+1} - (2)^{4n-2} \equiv (-1) - (2^2)^{2n-1} \equiv (-1) - (4)^{2n-1} \equiv (-1) - (1)^{2n-1} \equiv 0 \mod 5.$$

Problem 2. Prove the following divisibility statements.

(a) 4^{100} is not divisible by 3,

(b) $9|10^{n+1} + 9 \cdot n^2 + 4 \cdot 10^n - 5$,

Problem 3. Compute the last digit of the following numbers:

- 4^{100}, 2006^3, 923^{2006}, 7^{728}, and 9^{1234},
- Prove that the last digit of n^4, for any $n \in \mathbb{Z}$ must be $0, 1, 5$ or 6.

Solution: The last digits are $6, 6, 9, 1$ and 1. To prove these, we reduce modulo 10. First,

$$4^{100} \equiv (4^2)^{50} \equiv 16^{50} \equiv 6^{50} \equiv 6 \mod 10,$$

where we used the fact that $6^n \equiv 6 \mod 10$ for all positive integers n (check this by induction).

Similarly, $2006^3 \equiv 6^3 \equiv 6 \mod 10$.

Next,

$$923^{2006} \equiv 3^{2006} \equiv (3^4)^{501} \cdot 3^2 \equiv 81^{501} \cdot 9 \equiv 1 \cdot 9 \equiv 9 \mod 10.$$

Next, notice that $7^4 \equiv (7^2)^2 \equiv 49^2 \equiv 9^2 \equiv 1 \mod 10$, so

$$7^{728} \equiv (7^{182})^4 \equiv 1^4 \equiv 1 \mod 10.$$

Finally, $9^{1234} \equiv (9^2)^{616} \equiv 81^{616} \equiv 1^{616} \equiv 1 \mod 10$.

To prove that the last digit of n^4 is 0, 1, 5, or 6 for any integer n, it suffices to check one representative of each residue class modulo 10. Notice that we checked above that $7^4 \equiv 1 \mod 10$, and we remarked that $6^n \equiv 6 \mod 10$ for any positive integer n. Clearly $0^4 \equiv 0 \mod 10$ and $1^4 \equiv 1 \mod 10$. Finally,

$$\begin{aligned} 2^4 &\equiv 16 \equiv 6 \mod 10 \\ 3^4 &\equiv 81 \equiv 1 \mod 19 \\ 4^4 &\equiv 16^2 \equiv 6 \mod 10 \\ 5^4 &\equiv 25^2 \equiv 5^2 \equiv 5 \mod 10 \\ 8^4 &\equiv (8^2)^2 \equiv 4^2 \equiv 6 \mod 10 \\ 9^4 &\equiv 81^2 \equiv 1 \mod 10. \end{aligned}$$

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Problem 4. Compute the last two digits of the following numbers:

- 923^4, 1234998^{10}, and 2018^{2018}.

Solution: The last two digits are 41, 24 and 24. To prove these, we reduce modulo 100.

First,

$$923^4 \equiv 23^4 \equiv (23^2)^2 \equiv 529^2 \equiv 29^2 \equiv 841 \equiv 41 \mod 100.$$

Next,

$$1234998^{10} \equiv 98^{10} \equiv (-2)^{10} \equiv 1024 \equiv 24 \mod 100.$$

Finally, notice that

$$18^4 \equiv (18^2)^2 \equiv 324^2 \equiv 24^2 \equiv 576 \equiv 76 \mod 100,$$

so

$$2018^{2018} \equiv 18^{2018} \equiv (18^4)^{504} \cdot 18^2 \equiv 76^{504} \cdot 24 \equiv 76 \cdot 24 \equiv 1,824 \equiv 24 \mod 100,$$

where we used the fact that $76^n \equiv 76 \mod 100$ for all positive integers n (check this by induction).

Problem 5. Show that the following equations have no solution $x \in \mathbb{Z}$ over the integers:

(a) $9x^7 - 3x^2 + 2 = 0$,

(b) $15x^4 - 20x^2 + 56x^2 = 22$,

(c) $6x^9 - 7x^2 + 36 = 13$.

Solution: For Part (a) reduce modulo 3 to get

$$9x^7 - 3x^2 + 2 \equiv 0 \mod 3 \Longleftrightarrow 9x^7 - 3x^2 + 2 \equiv 0 \mod 3,$$

which has no solution modulo 3, and thus the original equation has no solution over the integers $\mathbb{Z}$. Another way to say it is that the left hand side of the equation give residue 2 when divide by 3 and the right hand side gives residue 0.

For Part (b) we reduce modulo 5 to get the equation

$$15x^4 - 20x^2 + 56x^2 \equiv 22 \mod 5 \Longleftrightarrow x^2 \equiv 2 \mod 5,$$

but if $x \in \mathbb{Z}$, then its square modulo 5 must be $x^2 \equiv 0, 1 \text{ or} 4 \mod 5$, and thus the equation has no solution modulo 5.

For Part (c) we work modulo 6 and obtain the equation

$$6x^9 - 7x^2 + 36 = 13 \mod 6 \Longleftrightarrow -x^2 \equiv 1 \mod 6 \Longleftrightarrow x^2 \equiv -1 \mod 6 \Longleftrightarrow x^2 \equiv 5 \mod 6.$$

However, the squares modulo 6 are $x^2 \equiv 0, 1, 3 \text{ or} 4 \mod 6$, and none of them is $-1 \equiv 5 \mod 6$.

Problem 6. Show that the following equations have no solutions $x, y \in \mathbb{Z}$ over the integers:

(a) $26x + 52y = 131$,

(b) $12x^3 - 3xy + 15y^2 = 1003$,

(c) $x^2 - 3y^2 = 15$,

(d) $x^4 + y^4 = 1599$.

Solution: For Part (a), the reduction modulo 13 gives

$$26x + 52y \equiv 131 \mod 13 \Longleftrightarrow 0 \equiv 1 \mod 13,$$

which is a contradiction. For Part (b) we reduce modulo 3, the left hand side becomes 0 and the right hand side 1, which proves there are no solutions modulo 3, and thus neither over the integers.

For Part (c), we reduce modulo 4 and we will get the equation

$$x^2 - 3y^2 \equiv 15 \mod 4 \Longleftrightarrow x^2 + y^2 \equiv 3 \mod 4,$$

but a square x^2 modulo 4 must have $x^2 \equiv 0, 1 \mod 4$, and thus $x^2 + y^2 \equiv 0, 1 \text{ or } 2$, but never 3.

For Part (d) reduce modulo 16 so that the right hand side becomes -1. Modulo 16 a fourth power is equivalent to 0 or 1, and so $x^4 + y^4 \equiv 0, 1 \text{ or } 2 \mod 16$, but $15 \equiv -1 \mod 16$, and thus it cannot be of the form $x^4 + y^4$ modulo 16.