Lecture 10: The Heisenberg uncertainty principle

Thm: (Heisenberg) Let \( A, B \in \mathcal{O}(V) \), and \( \mu \) a state. Then

\[ \mathcal{O}_\mu(A) \cdot \mathcal{O}_\mu(B) \geq \frac{\hbar}{2} |\mathcal{P}(\{A, B\})| \]  

Proof: First, \( \mathcal{O}_\mu(X) = \langle \mathcal{E}_\mu(X - \mathcal{E}_\mu(X)) \rangle \) is "convex" in the sense that if you pass the inequality for "pure states," then it follows for all states.

Second, the set of states is convex, its extreme points are called "pure states." For the class of examples \( \mathcal{G}(V) \in \mathcal{S}(V) \) the pure states are with \( \langle \rho | w \rangle = \langle v | w \rangle \) for \( v \in V \) and \( W \subseteq V \).

\[ \text{Projection operator } \mathcal{P}_\rho, \quad \text{"projective"} \]

Take, for a projection operator we'll have \( \mathcal{E}_\rho(A) = \langle A \rho, \sigma \rangle \).

Now, the proof: given \( A, B \) and \( \rho \), we know

\[ 0 \leq \|A^\ast AB \|= \langle A^\ast AB \rangle \leq \frac{1}{\hbar} \langle AB^\ast AB \rangle \langle \sigma | \sigma \rangle \]  

Since the \( \gamma \)-values of \( \alpha \) are \( \min \) of \( \frac{c - \frac{1}{\hbar} \langle AB^\ast AB \rangle \langle \sigma | \sigma \rangle}{c - \frac{1}{\hbar} \langle AB^\ast AB \rangle \langle \sigma | \sigma \rangle} \), \( (\#) \) implies that

\[ \langle A^\ast A \rangle - \frac{\hbar^2}{\langle AB^\ast AB \rangle} = 0 \quad \iff \quad \langle A^\ast A \rangle \leq \frac{\hbar^2}{\langle AB^\ast AB \rangle} = 0 \]  

Substitute \( A^\ast A \) by \( A - \mathcal{E}_\rho(A) \), same for \( B \). \( \iff \) this becomes \( \mathcal{O}_\mu(A) \cdot \mathcal{O}_\mu(B) \)

\( \Box \)