Lecture 24: Yang-Mills Field Theory (gauge theory) - contains its name for the old model, quantization still open.

**Ingredients:**
- (H, g) Ram manifold, G a Lie group (abelian/non-abelian)
- a principal G-bundle, A connection, FA curvature

**Def:** The Yang-Mills functional is:
\[ L_m(A) = \int \text{tr}(F_A \wedge F_A) \, d\mu_{\text{min}} \in \mathbb{R} \]
where A is a connection on a G-principal bundle. Each Lagrangian gives a set of motions. Always true: Branch.

The Yang-Mills eqn is:
\[ d_A(gF_A) = 0 \]
and \( d_A F_A = 0 \).

**B. Principal bundles:** G a Lie group and \( B \) (base) a smooth manifold. \( (B = M^4 \text{ for us}) \)

**Def:** A principal G-bundle is a smooth manifold \( P \) and a map \( \pi: P \rightarrow B \) such that \( \pi \) is locally trivial: \( \forall b \in B \exists \mathcal{U} \ni b \text{ open} \exists G \text{i.e. } \pi^{-1}(\mathcal{U}) \]

\[ (b, \rho) \mapsto \pi(b)(\rho) \]

**Rmk:** The class of all G-bundles depends solely on any top. \( H^1(B; \pi_1(G)) \) tells you enough:
\[ \text{eg. } H_1(U(1)) = H_1(\mathbb{R}) = \mathbb{Z}_2 \text{ for } \pi_1 \text{ of } \text{an } \text{U}(1) \text{-bundle on a } \text{sphere by } H^1(\mathbb{S}^2;\mathbb{Z}_2) = \mathbb{Z}_2 \]

Not take HRT25 and you can compute: 
(P has a section iff \( P = B \times G \)).

1. \( P \) has a section if \( P \) is a G-bundle.
2. If \( P \to G \text{U}(1) \) is a map, one can create a \( U(1) \text{-bundle} \).
   Difficult from \( \text{but} \)...

3. **Associated bundle:** \( P \times_V \text{ fiber by } V \).

**B.3. Parallel transport & Connections**

**Parallel Transport:** assign to each path \( Y \text{ from } b \text{ to } b' \in B \) an isomorphism \( P_Y: G \rightarrow G \)

\[ \pi_1(Y) \times \pi_1(Y) \]

\( \text{same with } v, v' \).

**Connection:** The idea is to give data (a \( g \)-valued 1-form) such that \( P_Y \) is given by solving an ODE for the data.

\[ A \in \mathfrak{g} \text{-valued } 1 \text{-form} \]

\[ \text{how to identify true fibers?} \]

\[ \pi_1(Y) \times \mathfrak{g} \]

\( \pi_1(Y) \times \mathfrak{g} \) is an exact sequence.

**Connection:** the idea is to give data (a \( g \)-valued 1-form) such that \( P_Y \) is given by solving an ODE for the data.

\[ A \in \mathfrak{g} \text{-valued } 1 \text{-form} \]

\[ \text{The connection equation for } A \text{ is} \]
\[ d_Ai = d + A \]

\[ \text{Pf} \text{y is obtained by solving ODE} \]
\[ d_A = 0. \text{ iff flat connection} \]