Lecture 26

Tuesday, September 29, 2020 12:05 PM

Lecture 26: Donaldson Theory (or how to use gauge thy in topology)

Yang-Mills: P ppal G-bundle ($SU(2)$, $U(1)$), $P \longrightarrow B$ smooth base. A a connection: $L_{YM}(A) = \int_{M^4} F_A \wedge * F_A \in \mathbb{R}$

} EL

YM Eq^n: $\boxed{* d_A * F_A = 0}$ (closed) $\quad (d_A F_A = 0) \rightsquigarrow \mathcal{M}_{YM} := \dfrac{\{\text{all YM connections}\}}{\text{gauge gp } \mathcal{G}}$ ← a space!

Example: $G = U(1)$ then a form A defines a Yang-Mills connection if $*(d+A)*(d+A)^2 = 0$.

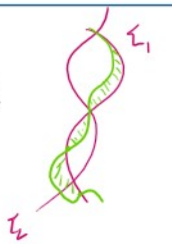
In general $(d+A)^2 = \underbrace{dA}_{F_A} + A \wedge A$, for $G = U(1)$ we get $A \wedge A = 0$ ⇝ harmonicity $\quad M \simeq (S^1)^{b_1(M)}$

↳ Laplacian $\Delta \subset \Omega^\bullet$ is defined $\Delta := d(*d*) + d(*d*) \subset \Omega^\bullet$ (adjoint of d), k-form is harmonic if $\Delta\omega = 0$.

Thm: (Hodge) For any $a \in H^k(M;\mathbb{R})$ $\exists! \omega \in \Omega^k$ harmonic and $[\omega] = a$. ⇝ $\boxed{\dfrac{H^1(M;\mathbb{R})}{[M \to S^1]}}$ ← harmonic 1-forms

$H^1(M;\mathbb{Z})$

§2. 4-manifolds: S^4, $S^2 \times S^2$, $\mathbb{CP}^2$, $S^1 \times S^3$, T^4.



M^4 4-manifold: $\Sigma_1, \Sigma_2 \subseteq M^4$ surfaces then $\Sigma_1 \cap \Sigma_2$ fte. (generic intersection).

$Q_M : H_2(M;\mathbb{Z}/_{tors}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$

bilinear form

classical topology (Casson)

$\boxed{\text{Thm: (Freedman) Given any bilinear form } Q, \exists M \text{ 4-manifold such that } Q_M = Q. \longrightarrow \text{many examples of topol. 4-folds!} \quad \text{only topological in his proof}}$

Yang Mills →

$\boxed{\text{Thm. (Donaldson's Diag. Thm.) Let } M^4 \text{ be a SMOOTH 4-manifold simply-connected s.t. } Q_M \text{ is positive definite. Then } Q_M \text{ is diagonalizable; } \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}$

§3. Scheme of Donaldson's proof: (M^4, g) generic and consider $P \longrightarrow M^4$, if $G = SU(2)$.

$\boxed{\text{Ex.: } P \xrightarrow{SU(2)} S^4 ? \quad \rightsquigarrow \text{clutching is } S^3 = \partial D^4 \longrightarrow SU(2) \quad \pi_3(SU_2) = \pi_3(S^3) = \mathbb{Z}. \quad D^4 \times SU(2) \longrightarrow D^4}$

the classification is understood ⇝ $c_2(P) \in H^4(M;\mathbb{Z})$ determines P

↓ CHOOSE $c_2(P) = 1$!

Consider the moduli of Yang Mills connections which are ASD: $\boxed{-F_A = * F_A}$

(i) Dimension of $\mathcal{M}_{ASD}(M^4; SU(2); g)$ is given by Atiyah-Singer ← computes ind(D) of D Fredholm via topology $\text{ind}(D) = \underbrace{\int_M \text{top.}}_{\text{algebraic topology}}$ (analysis)

for this $\dim \mathcal{M}_{ASD} = 8 \cdot c_2(P) - 3 \overset{c_2=1}{=} 5$.

(ii) The moduli $\mathcal{M}_{ASD}(SU(2))$ looks like

↪ $\boxed{\mathcal{M}_{ASD} \text{ cobordism between } \#\mathbb{CP}^2\text{'s and } M^4 !}$