Lecture 3

Tuesday, September 29, 2020 12:05 PM

# Lecture 3 : Hamiltonian Formalism

Compare $H = K + \mathcal{U}$ to $L = K - \mathcal{U}$

The dynamics are on the cotangent bundle $T^*M \xrightarrow{\pi} M$.

Locally, $(q_1,\dots,q_n)$ coordinates in a chart $\mathcal{U} \subseteq M$, $\mathcal{U}$ a ball $D^n$

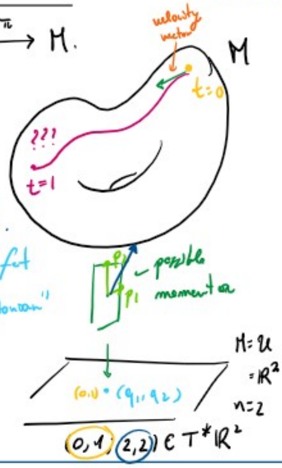
and $(p_1,\dots,p_n)$ coordinates in $T^*\mathcal{U}$ in fiber of $\pi$ direction.

Dynamics are obtained by choosing a $H: T^*M \longrightarrow \mathbb{R}$. choice we make, $C^\infty$-fct "the Hamiltonian"

↳ trajectories will be integral curves of a vect. field — generalize E-L eq$^n$

"the Hamiltonian v.f." $X_H$

today's goal!




## §1. Symplectic Geometry of $T^*M$

: adding a sympl. form $\omega$ in $T^*M$ will give the transition $H \rightsquigarrow X_H$.

"think of gradient v.f. $\langle \nabla f, \cdot\rangle_g = df$" of $f$ — before / for metric

There are two canonical forms $\lambda \in \Omega^1(T^*M)$ (Liouville form),

symplectic form $\to d\lambda = \omega \in \Omega^2(T^*M)$.

$\alpha^*(☺) = ☺ \circ \alpha$, ☺

$\alpha: M \to T^*M$, $p \mapsto \alpha_p$ covector in $T_p^*M$

Chap 2.2 Ana Cannas book on Symp Geom

- Def$^n$: $\lambda \in \Omega^1(T^*M)$ is the unique 1-form satisfying $\alpha^*\lambda = \alpha$ for any $\alpha \in \Omega^1(M)$ ∎

(*) Ex: Locally $M = \mathbb{R}^n$, $\lambda = \sum_{i=1}^n p_i\, dq_i$ ($p_i \to f_i$). Given $\alpha = \sum_{i=1}^n f_i(q)\, dq_i$, $\alpha^*\lambda = \left(\sum p_i\, dq_i\right)\Big|_{p_i \to f_i(q)}$ ∎

- Def$^n$: $\omega \in \Omega^2(T^*M)$ is $\omega = -d\lambda$. (*)

Locally $\omega = -d\lambda = -d\left(\sum_{i=1}^n p_i\, dq_i\right) = \sum_{i=1}^n -dp_i \wedge dq_i = \sum_{i=1}^n dq_i \wedge dp_i$.

**Prop**: (i) $\omega \in \Omega^2(T^*M)$ is closed ($d\omega = 0$) and non-deg ($\omega^n \neq 0$ if $n = \dim M$; 2n-form ⇔ $\omega^n$ volume form) } this is called a symplectic form.

(ii) Given a Hamiltonian $H: T^*M \longrightarrow \mathbb{R}$

$\exists!\ X_H \in \Gamma(T(T^*M))$ (sections) a v.f. on $T^*M$ such that

$$(\#)\quad \boxed{\omega(X_H, -) = dH}$$

morally, it is a system where you can invert $\omega$ b/c $\det(\omega) \neq 0$.

**Proof**: (i) can be verified locally: $\omega = \sum_{i=1}^n dq_i dp_i$ then $d\omega = 0$ and $\omega^n = dq_1 dp_1 \dots dq_n dp_n \neq 0$.

(ii) Locally we have the system: $\left(\sum_{i=1}^n dq_i dp_i\right)\left(a_1\partial_{q_1} + \dots + a_n\partial_{q_n} + b_1\partial_{p_1} + \dots + b_n\partial_{p_n}\right) = \sum_{i=1}^n \partial_{q_i}H\, dq_i + \partial_{p_i}H\, dp_i$.

The Haml. v.f. $X_H = \sum_{i=1}^n a_i\partial_{q_i} + b_i\partial_{p_i}$ is defined by $\quad \omega = \sum_{i=1}^n dq_i dp_i$

$$\omega(X_H, \cdot) = \sum_{i=1}^n a_i\, dp_i - b_i\, dq_i \overset{(\#)}{=} \sum_{i=1}^n (\partial_{q_i}H)\, dq_i + (\partial_{p_i}H)\, dp_i =: dH$$

$\Rightarrow$ (*) $a_i = \partial_{p_i}H$, $i \in [1,n]$; $b_i = -\partial_{q_i}H$, $i \in [1,n]$ } local sol$^n$ exists $\Longrightarrow$ global sol$^n$ exists. ∎

Prop gives that a trajectory $\gamma: [a,b] \longrightarrow T^*M$ is integral curve $\overset{\text{locally}}{\Longleftrightarrow}$

$\gamma(t) = (q_i(t), p_i(t))$, $\dot\gamma(t) = (\dot q_i(t), \dot p_i(t))$

$$\boxed{\begin{aligned}\dot p_i(t) &= -\partial_{q_i}H\\ \dot q_i(t) &= \partial_{p_i}H\end{aligned}}$$

Hamilton's equations 1st order!