MAT 265: PROBLEM SET 2

DUE TO FRIDAY OCT 16 AT 9:00AM

Abstract. This is the second problem set for the graduate course Mathematical Quantum Mechanics in the Fall Quarter 2020. It was posted online on **FILL IN** and is due Friday Oct 16 at 9:00am via online submission.

Purpose: The goal of this assignment is to review and practice *symmetries* from Mathematical Quantum Mechanics (MAT265). In particular, we would like to become familiar with many examples of Hamiltonian systems, including Hamiltonian vector fields, as well as with the infinitesimal symmetries of Lagrangians and their corresponding constants of motion.

Task and Grade: Solve two of the six problems below. Each Problem is worth $33.\hat{3}$ points.The maximum possible grade is 100 points. Despite the task being three problems, I strongly encourage you to work on the six problems.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Textbook: We will use "A Brief Introduction to Physics for Mathematicians" by I. Dolgachev. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. (Poisson Brackets) Let $M = \mathbb{R}^n$ and $f, g \in C^\infty(T^*M)$. Consider the function

$$\{f, g\} := \sum_{i=1}^{n} \partial_{q_i} f \cdot \partial_{p_i} g - \partial_{p_i} f \cdot \partial_{q_i} g.$$

(i) Prove that $\{\cdot, \cdot\}$ is a Poisson bracket: bilinear, antisymmetric and satisfies by Jacobi's identity and Leibniz's rule.

(ii) Show that $\{f, g\} = \omega(X_f, X_g)$, where X_h is the Hamiltonian vector field of h.

(iii) Show that $X_{\{f,g\}} = -[X_f, X_g]$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields.

(iv) (Another Poisson bracket) Consider $\mathbb{R}^3$ with the cross product $\times$. Define the operation

$$\{\cdot, \cdot\}_{\mathfrak{so}(3)} : C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3) \longrightarrow C^\infty(\mathbb{R}^3),$$
$$\{f, g\}_{\mathfrak{so}(3)}(q_1, q_2, q_3) := -(q_1, q_2, q_3) \cdot (\mathrm{grad}(f) \times \mathrm{grad}(g)).$$

Show that $\{\cdot, \cdot\}_{\mathfrak{so}(3)}$ is also a Poisson bracket on $\mathbb{R}^3$.

There are many interesting brackets out there, useful for solving PDEs, doing geometry and probability, and more: e.g. see the Ideal Fluid Bracket, the KdV Bracket, the Poisson-Vlasov Bracket or Toda Lattice Bracket.

Problem 2. (Example 4 in I. Dolgachev's Notes) Consider the hyperbolic plane $M = \mathbb{H} = \{(q_1, q_2) \in \mathbb{R}^2, q_2 > 0\}$ with its (constant negative curvature) metric

$$g = \frac{1}{q_2^2} \left(dq_1 \otimes dq_1 + dq_2 \otimes dq_2 \right),$$

and the Lagrangian $L = g$. Show that the three vector fields

$$\eta_1 = q_1 \partial_{q_1} + q_2 \partial_{q_2}, \quad \eta_2 = (q_2^2 - q_1^2)\partial_{q_1} - 2q_1 q_2 \partial_{q_2}, \quad \eta_3 = \partial_{q_1},$$

are infinitesimal symmetries for L, and compute their associated constants of motion. (Optional: what is the Lie algebra generated by η_1, η_2, η_3 ? Relate it to the group $\mathrm{PSL}_2(\mathbb{R})$ of real Möbius transformations and traceless (2×2)-matrices.).

Problem 3. (Infinitesimal Symmetries of Lagrangians) In this problem, $L : TM \longrightarrow \mathbb{R}$ is a Lagrangian, $\eta \in \Gamma(TM)$ a vector field on M and, if η is a symmetry for L, I_η is the associated constant of motion for the dynamical system defined by L. (Drawing a picture can be helpful to understand symmetries).

(a) Let $M = \mathbb{R}^2 \setminus \{0\}$ have Cartesian coordinates (q_1, q_2), $r = q_1^2 + q_2^2$, and

$$L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(r).$$

Consider the vector field $\eta = q_1 \partial_{q_2} - q_2 \partial_{q_1}$. Show that η is an infinitesimal symmetry of L and find its associated constant of motion $I_\eta : TM \longrightarrow \mathbb{R}$.

(b) Consider a particle of mass m moving in $M = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}_z$, with cylindrical coordinates $(r, \phi, z) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}_z$. Suppose the potential energy $V = V(r, z + \phi)$ of the system only depends on the coordinates $\phi + z$ and r.

Find an infinitesimal symmetry η of $L = K - V(r)$, where K is the kinetic energy, such that $I_\eta = m(r^2\dot{\phi} - \dot{z})$.

(c) Consider a particle moving in $M = \mathbb{R}^3$ with a potential energy $V = V(r)$ which only depends on $r = q_1^2 + q_2^2 + q_3^2$. Show that the three vector fields

$$\eta_1 = q_1\partial_{q_2} - q_2\partial_{q_1}, \quad \eta_2 = q_2\partial_{q_3} - q_3\partial_{q_2}, \quad \eta_3 = q_3\partial_{q_1} - q_1\partial_{q_3},$$

are infinitesimal symmetries for such a Lagrangian system. Find the associated constants of motion, and try to describe the physical meaning of the vector $(I_{\eta_1}, I_{\eta_2}, I_{\eta_3})$.

(d) Generalizing (a) and (c), consider a particle moving in $M = \mathbb{R}^n$ with a potential energy $V = V(r)$ which only depends on $r = \sum_{i=1}^n q_i^2$, i.e. a rotationally symmetric potential energy. Find $d = n(n-1)/2$ infinitesimal symmetries $\eta_i \in TM$, $i \in [1, d]$, of the associated Lagrangian.[1]

Problem 4. (Hamiltonian Vector Fields in the Circle)[2] For each of the examples below, M is the configuration space M, $H : T^*M \longrightarrow \mathbb{R}$ the Hamiltonian and X_H the associated Hamiltonian vector field on T^*M.

(a) Consider a free particular in a circle $M = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, with coordinate $\theta \in \mathbb{S}^1$. The Hamiltonian is

$$H(\theta, p) = \frac{1}{2m}p^2,$$

where $(\theta, p) \in T^*\mathbb{S}^1 = \mathbb{S}^1_\theta \times \mathbb{R}_p$. Find X_H and describe its trajectories in $T^*\mathbb{S}^1$.

(b) Consider a planar pendulum of length l and mass m, with $M = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ a circle, with coordinate $\theta \in \mathbb{S}^1$. The Hamiltonian is

$$H(\theta, p) = \frac{1}{2m}p^2 + mg\sin(\theta),$$

where $(\theta, p) \in T^*\mathbb{S}^1 = \mathbb{S}^1_\theta \times \mathbb{R}_p$. Draw the trajectories of the vector field X_H in $T^*\mathbb{S}^1$ and interpret them geometrically, in terms of the position and momentum of the pendulum.

(c) Let $M = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Does there exist a Hamiltonian H such that $X_H = \partial_p$?

(d) Consider $M = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and an arbitrary smooth Hamiltonian

$$H(\theta, p) = H(\theta) : \mathbb{S}^1 \longrightarrow \mathbb{R}.$$

Let $\mathfrak{S} = \{(\theta, p) \in T^*\mathbb{S}^1 : p = 0\}$ be the space of all the possible initial positions for a particle at rest. Show that the t-flow $\varphi_H^t : T^*\mathbb{S}^1 \longrightarrow T^*\mathbb{S}^1$ of X_H *always* has at least two fixed points.[3] (Geometrically, this says that the intersection $\varphi_H^t(\mathfrak{S}) \cap \mathfrak{S}$ contains at least two points *for all* $t \in \mathbb{R}^+$.)

[1] These d infinitesimal symmetries must be distinct, that is, they should span a d-dimensional vector subspace of T_qM at each point $q \in \mathbb{R}^n$ which is not the origin $r = 0$.

[2] Equivalently, for a line $M = \mathbb{R}$ with periodic boundary conditions.

[3] Thus, *any* Hamiltonian system of a particle in a circle always has at least two initial positions for a particle at rest such that, once the particle is placed there, it does not move as time evolves.

(e) (Optional) Consider a particle moving in a 2π-unit square with periodic boundary conditions, so $M = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}^2_{q_1,q_2}/(q_1 \sim q_1 + 2\pi, q_2 \sim q_2 + 2\pi)$ is a 2-torus. Show that a vector field

$$X = A_1\partial_{q_1} + A_2\partial_{p_1} + A_3\partial_{q_2} + A_4\partial_{p_2}, \quad A_i \in C^\infty(T^*\mathbb{T}^2),$$

is a Hamiltonian vector field $X = X_H$ for a function $H : T^*\mathbb{T}^2 \longrightarrow \mathbb{R}$ iff

$$\int_{\gamma_i} (A_1 dp_1 - A_2 dq_1 + A_3 dp_2 - A_4 dq_2) = 0, \quad i = 1, 2,$$

where the curves γ_1, γ_2 are

$$\gamma_1 = \{(q_1, q_2, p_1, p_2) \in T^*\mathbb{T}^2 : q_2 = p_1 = p_2 = 0\},$$

$$\gamma_2 = \{(q_1, q_2, p_1, p_2) \in T^*\mathbb{T}^2 : q_1 = p_1 = p_2 = 0\}.$$

Problem 5. (Some Real Life Systems with Symmetries)

(a) (Kepler Problem) The Hamiltonian two-body problem is given by $M = \mathbb{R}^3$ and the Hamiltonian

$$H(q, p) = \frac{m}{2}(p_1^2 + p_2^2 + p_3^2) + V(r), \quad V \in C^\infty(\mathbb{R}),$$

where $r = q_1^2+q_2^2+q_3^2$ is the distance to the origin.[4] Consider the three functions

$$I_1 = q_2p_3 - p_2q_3, \quad I_2 = q_3p_1 - p_3q_1, \quad I_3 = q_1p_2 - q_2p_1.$$

Show that I_3 and $I_1^2 + I_2^2 + I_3^2$ are constants of motion for the Hamiltonian system, and prove that they are in involution, that is:

$$\{I_3, I_1^2 + I_2^2 + I_3^2\} = 0.$$

(b) (Euler Tops)[5] Consider a rotating rigid body in $\mathbb{R}^3$ fixed at a point but is not subject to any external forces (like an ideal spinning top). Let $i_1, i_2, i_3 \in \mathbb{R}$ be the moments of inertia of the rigid body, then $M = \mathbb{R}^3$ and the Hamiltonian is the kinetic energy

$$H(q, p) = \frac{I_1^2}{2i_1} + \frac{I_2^2}{2i_2} + \frac{I_3^2}{2i_3},$$

where the functions $J_i \in \mathbb{C}^\infty(\mathbb{R}^3)$ are defined in Part (a). Find *four* constants of motions for this Hamiltonian system.

(c) (Spherical Pendulum) Let $(\theta, \varphi) \in \mathbb{S}^2$ be spherical coordinates for $M = \mathbb{S}^2 = \{(q_1, q_2, q_3) \in \mathbb{R}^3 : q_1^2 + q_2^2 + q_3^2 = 1\}$, φ the azimuth and θ the altitude. This is the configuration space of a spherical pendulum. Suppose that the pendulum has mass $m = 1$, length $l = 1$, and it is subjected to a gravitational potential given by the restriction of $z : \mathbb{R}^3 \longrightarrow \mathbb{R}$ to $\mathbb{S}^2$. First, show that the Hamiltonian of this system is

$$H(\theta, \varphi; p_\theta, p_\varphi) = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) - g\cos\theta,$$

[4] See Problem 1.(c) for the same initial scenario. The classical planetary motion problem studied by J. Kepler (1610s) had radial potential $V(r) = r^{-1}$.

[5] For more fun, learn about Lagrange tops (gyroscopes) and Kovalevskaya tops.

where $H : T^*\mathbb{S}^2 \longrightarrow \mathbb{R}$ and p_θ, p_φ are (local) coordinates on the cotangent fiber of a point (θ, φ). Then, find a constant of motion distinct from H.

Problem 6. (Geodesic Systems with Symmetries) Let us consider a free particle of mass $m = 1$ moving on a surface S. Different Riemannian metrics (S, g) on that surface yield different models for the kinetic energy of the particle, and thus its motion. Since some metrics g are more symmetric than others, certain symmetric choices of g will give geodesics with interesting constants of motions.

(a) (Clairaut's Integral) Consider a surface of revolution $S \subseteq \mathbb{R}^3$ with its Riemannian metric induced from the flat metric $(\mathbb{R}^3, g_{\text{st}})$, $(r, \phi, z) \in \mathbb{R}^3$ cylindrical coordinates and z the axis of revolution. Show that $r^2\dot{\phi}$ is a constant of motion.

(b) (Funky, but somewhat symmetric, metrics on $\mathbb{T}^2$) Consider the 2π-periodic coordinates $(q_1, q_2) \in \mathbb{T}^2$ and the metric

$$g_{\mathbb{T}^2} = (f(q_1) + g(q_2))(dq_1 \otimes dq_1 + dq_2 \otimes dq_2), \quad f, g \in C^\infty(\mathbb{T}^2),$$

and suppose $f(q_1) + g(q_2) \neq 0$. Show that the function

$$J : T^*\mathbb{T}^2 \longrightarrow \mathbb{R}, \quad J(q_1, q_2, p_1, p_2) = \frac{g(q_2)p_1^2 - f(q_1)p_2^2}{f(q_1) + g(q_2)}$$

is a constant of motion for the geodesic system with Lagrangian $L = g_{\mathbb{T}^2}/2$.

(c) (Geodesic System on Ellipsoids) Let $a_1, \ldots, a_n \in \mathbb{R}^+$ be such that

$$a_n < a_{n-1} < \ldots < a_2 < a_1.$$

Consider the $(n-1)$-dimensional ellipsoid

$$E(A) = \{q = (q_1, \ldots, q_n) \in (\mathbb{R}^n, g_{\text{st}}) : \langle q, A^{-1}q\rangle = 1\} \subseteq \mathbb{R}^n,$$

where $A := \operatorname{diag}(a_1, \ldots, a_n)$ is a diagonal matrix. Consider the Lagrangian $L = g_{\text{st}}|_{E(A)}/2$. Show that the n functions

$$J_l(q, \dot{q}) = \dot{q}_l + \sum_{k=1, k\neq l}^{n} \frac{(q_l\dot{q}_k - q_k\dot{q}_l)^2}{a_l - a_k}.$$

are constants of motion.