MAT 265: PROBLEM SET 3

DUE TO FRIDAY OCT 30 AT 9:00AM

Abstract. This is the third problem set for the graduate course Mathematical Quantum Mechanics in the Fall Quarter 2020. It was posted online on **Oct 16** and is due Friday Oct 30 at 9:00am via online submission.

Purpose: The goal of this assignment is to review and practice *Poisson brackets* and *Lie algebras* from Mathematical Quantum Mechanics (MAT265). In particular, we would like to become familiar with many examples of Poisson structures, including those in the dual Lie algebra $\mathfrak{g}^*$.

Task and Grade: Solve two of the six problems below. Each Problem is worth 50 points.The maximum possible grade is 100 points. Despite the task being three problems, I strongly encourage you to work on all the problems.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Textbook: We will use "A Brief Introduction to Physics for Mathematicians" by I. Dolgachev. Please contact me *immediately* if you have not been able to get a copy of any edition.

Problem 1. (Poisson Brackets and Cross Products) Let us consider $(\mathbb{R}^3, \times)$, where $\times$ is the cross product, and $(\mathfrak{so}(3), [\cdot,\cdot])$ the space of (3×3)-skew-symmetric matrices

$$\mathfrak{so}(3) := \{A \in M_3(\mathbb{R}) : A^t + A = 0\}$$

endowed with the bracket $[A, B] = AB - BA$, $A, B \in \mathfrak{so}(3)$.

(i) Prove that $(\mathbb{R}^3, \times)$ is a Lie algebra, i.e. $v \times w$ is bilinear, skew-symmetric and satisfies the Jacobi identity.

(ii) Prove that $(\mathfrak{so}(3), [\cdot,\cdot])$ is a Lie algebra, i.e. $\mathfrak{so}(3)$ is closed under $[\cdot,\cdot]$, and the bracket $[\cdot,\cdot]$ is bilinear, skew-symmetric and satisfies the Jacobi identity.

(iii) Show that the map

$$\varphi : (\mathbb{R}^3, \times) \longrightarrow (\mathfrak{so}(3), [\cdot,\cdot]), \quad \varphi(x_1, x_2, x_3) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

is an isomorphism of Lie algebras, i.e. φ is an isomorphism of $\mathbb{R}$-vector spaces and it is a Lie algebra morphism: $\varphi(v \times w) = [\varphi(v), \varphi(w)]$.

(iv) Consider $\mathfrak{g}^* = \mathfrak{so}(3)^*$ and note that $x_1, x_2, x_3 \in \mathfrak{g}^*$. Let $f, g \in C^\infty(\mathfrak{so}(3)^*)$. Show that the bracket

$$\{f,g\}(x_1,x_2,x_3) = \det \left\| \begin{matrix} x_1 & x_2 & x_3 \\ \partial_{x_1} f & \partial_{x_2} f & \partial_{x_3} f \\ \partial_{x_1} g & \partial_{x_2} g & \partial_{x_3} g \end{matrix} \right\|$$

is a Poisson bracket on $\mathfrak{g}^* = \mathfrak{so}(3)^*$.

Hint: One can certainly compute directly. A more conceptual approach is to show that the right-hand side of the equality is actually[1] $x([df_0, dg_0])$, where $0, x \in \mathfrak{so}(3)^*$. Then the necessary properties follow from *(ii)*.

(v) Consider $I_1, I_2, I_3 \in \mathbb{R}^+$, $I_1 < I_2 < I_3$, and the functions $f, g \in C^\infty(\mathfrak{so}(3)^*)$:

$$f(x_1,x_2,x_3) = \frac{x_1^2}{2I_1} + \frac{x_1^2}{2I_2} + \frac{x_3^2}{2I_3}, \quad g(x_1,x_2,x_3) = x_1^2 + x_1^2 + x_3^2.$$

Compute the matrices $df_0, dg_0 \in \mathfrak{so}(3)$ and their Lie bracket $[df_0, dg_0]$. Relate the resulting computation to the constants of motion of a rigid body.

(vi) (Optional) In the article "On teaching mathematics", V.I. Arnol'd asserts "The Jacobi identity (which forces the heights of a triangle to cross at one point) is an experimental fact [...]". Show that the Jacobi identity for $(\mathbb{R}^3, \times)$ implies that the three altitudes of a triangles intersect at *exactly one* point.[2]

Problem 2. (The Spherical Pendulum). In this problem we study the *spherical pendulum*, a mass $m = 1$ moving friction-less on the surface of the 2-sphere

$$\mathbb{S}^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The only force acting is gravity and the constraint to $\mathbb{S}^2$. The Hamiltonian is

$$H(x,y,z,p_x,p_y,p_z) = K + U = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + z,$$

where $(x,y,z,p_x,p_y,p_z) \in T^*\mathbb{R}^3$ are coordinates in $T^*\mathbb{R}^3$. The Hamiltonian system is thus given by $(T^*\mathbb{S}^2, H|_{T^*\mathbb{S}^2})$, with configuration space $\mathbb{S}^2$ and phase space $T^*\mathbb{S}^2$.

(i) Show that $J = xp_y - yp_x$ is a conserved quantity and give a physical interpretation of J. Deduce that the motion of a particle in phase space must occur on the fiber of the smooth map[3]

$$\mu := (H, J) : T^*\mathbb{S}^2 \longrightarrow \mathbb{R}^2.$$

(ii) Consider the curve $\tau \subseteq \mathbb{R}^2$ given as the image of

$$\tau : \mathbb{R} \setminus (-1,1) \longrightarrow \mathbb{R}^2, \quad \tau(s) = \left(\frac{s^2}{2} - \frac{3}{s^2}, s - \frac{1}{s^3}\right).$$

Show that the image of $\mu(T^*\mathbb{S}^2)$ is given by the closure of the *upper* connected component of $\mathbb{R}^2 \setminus \tau$, i.e. the region equal or above the curve $\tau \subseteq \mathbb{R}^2$.

[1]Note that df_0 is a linear function on $T_0\mathfrak{g}^* \cong \mathfrak{g}^*$, thus canonically an element of $\mathfrak{g} \cong \mathfrak{g}^{**}$.
[2]This is called the orthocenter of the triangle.
[3]This is an example of a *momentum map*.

(iii) Describe the physical motion corresponding to the two level sets $\mu^{-1}(1,0)$ and $\mu^{-1}(-1,0)$. What is particular to the physical motion corresponding to the level sets of the form $\mu^{-1}(h,0)$?

(iv) Study the fibers of the map μ, i.e. the level sets $\mu^{-1}(c)$, where $c \in \mathbb{R}^2$.

(v) Describe the 3-space given as the pre-image $\mu^{-1}(\mathbb{R} \times \{0\})$.

(vi) (Optional) Describe the 3-space given as the pre-image $\mu^{-1}(\mathbb{R} \times \{2\})$.

Problem 3. (An integrable system discovered in 2017)[4] The phase space of the Hamiltonian system is the symplectic manifold[5]

$$\mathbb{S}^2 \times \mathbb{S}^2 := \{(x_1, y_1, z_1; x_2, y_2, z_3) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_1^2 + y_1^2 + z_1^2 = 1, x_2^2 + y_2^2 + z_2^2 = 1\},$$

and symplectic form $\omega = \omega_{\text{st}} \oplus 2\omega_{\text{st}}$ where ω_{st} is the area-1 2-form, i.e. a volume form, on $\mathbb{S}^2$. This can be described as the restriction of the volume form $\omega = dx_1dy_1dz_1 \oplus 2(dx_2dy_2dz_2)$ of $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{S}^2 \times \mathbb{S}^2$. The Hamiltonian of the system is

$$H(x_1, y_1, z_1; x_2, y_2, z_3) := \frac{z_1 + z_2}{4} + \frac{1}{2}(x_1x_2 + y_1y_2).$$

(1) Show that $J = z_1 + 2z_2$ is a constant of motion.

(2) Consider the moment map $\mu = (H, J) : \mathbb{S}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$. Show that μ has *exactly* four singular points at $(p_N, p_N), (p_S, p_N), (p_N, p_S), (p_S, p_S)$, where $p_N, p_S \in \mathbb{S}^2$ are the North and South poles $p_N = (0, 0, 1), p_S = (0, 0, -1)$.

(3) Give a sketch of the image $im(\mu) \subseteq \mathbb{R}^2$.

(4) Show that the regular level sets of $H : \mathbb{S}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ are topologically either $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$. (*Hint*: Use the map μ and its image $im(\mu)$ as studied in (2) and (3).)

(5) Prove that $H^{-1}(0)$ is not a regular level set. Discuss the topology of regular level sets $H^{-1}(\varepsilon)$, $H^{-1}(-\varepsilon)$, right above and below $H^{-1}(0)$, $\varepsilon \in \mathbb{R}^+$ small.

(6) (Optional) Explain why (5) does not contradict one of the principles of Morse theory: crossing a singular level set of a generic function $f : M \longrightarrow \mathbb{R}$ with a unique critical point at that level set must give a different homotopy type for the level sets above than that of the level sets below.

[4]It appeared in the article "A family of compact semitoric systems with two focus-focus singularities", by S. Hohloch, J. Palmer in the *Journal of Geometric Mechanics*, 2018, 10(3): 331-357.

[5]Note that this is *not* a cotangent bundle, but rather it comes from taking a quotient of a certain magnetic cotangent bundle $T^*\mathbb{S}^3 \cong \mathbb{S}^3 \times \mathbb{S}^2$ by $\mathbb{S}^1$-symmetries, through a process called *symplectic reduction*.

Problem 4. (The Lotka-Volterra Equations are Hamiltonian) In population dynamics, we consider the populations $x_1(t), \dots, x_n(t) \in \mathbb{R}^+$ of n species at a given time $t \in \mathbb{R}_{\geq 0}$. The Lotka-Volterra predator-prey model (1910 & 1925) gives the following system of n non-linear differential equations

$$x_i'(t) = \alpha_i x_i(t) + \sum_{j=1}^{n} \beta_{ij} x_i(t) x_j(t).$$

Suppose that the matrix $(\beta_{ij}) \in M_n(\mathbb{R})$ is skew-symmetric and invertible.

(i) Let $f_1, f_2 \in C^\infty(\mathbb{R}^n)$. Show that

$$\{f_1, f_2\} := \sum_{i<j} \beta_{ij} x_i x_j \left(\partial_{x_i} f_1 \cdot \partial_{x_j} f_1 - \partial_{x_i} f_2 \cdot \partial_{x_j} f_1 \right)$$

is a Poisson bracket.

(ii) Find a Hamiltonian $H \in C^\infty(\mathbb{R}^n)$ such that the Lotka-Volterra system of differential equations becomes

$$x_i'(t) = \{x_i(t), H\}, \quad i \in [1, n].$$

Hint: Try with a linear combination of x_i and $\ln(x_i)$, $i \in [1, n]$.

Problem 5 (Classical Lie Groups G and Their Lie algebras $\mathfrak{g}$) Let us consider the following four matrix Lie groups:

$$GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det(A) \neq 0\}, \quad SL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det(A) = 1\},$$

$$SO(n, \mathbb{R}) := \{A \in M_{n\times n}(\mathbb{R}) : \langle Av, Aw \rangle = \langle v, w \rangle, \forall v, w, \in \mathbb{R}^n\} = \{A \in M_{n\times n}(\mathbb{R}) : A^t A = Id_n\},$$

$$Sp(2n, \mathbb{R}) := \{A \in M_{2n\times 2n}(\mathbb{R}) : \omega(Av, Aw) = \omega(v, w), \forall v, w, \in \mathbb{R}^n\} = \{A \in M_{n\times n}(\mathbb{R}) : A^t \Omega A = \Omega\},$$

$$\text{where } \Omega = \begin{pmatrix} 0 & Id_n \\ Id_n & 0 \end{pmatrix}.$$

Geometrically, $GL(n, \mathbb{R})$ are the linear automorphisms of $\mathbb{R}^n$ as a vector space. The group $SL(n, \mathbb{R})$ consists of those automorphisms which preserve volume and orientation. The group $SO(n, \mathbb{R})$ consists of those automorphisms which preserve the standard inner product (the dot product), and the group $Sp(n, \mathbb{R})$ consists of those automorphisms which preserve the standard symplectic form ω_{st}, equivalently the skew-symmetric matrix Ω.

(a) For each of the Lie groups G above, describe the vector space $\mathfrak{g} := T_e G$.

(b) For each of the four example above, verify that $\mathfrak{g}$ is closed under the matrix commutator bracket:

$$[\cdot, \cdot] : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R}), \quad [A, B] := AB - BA.$$

The pair $(\mathfrak{g}, [\cdot, \cdot])$ is called a Lie algebra, as one can verify that $[A, B]$ is bilinear, skew-symmetric and satisfies the Jacobi identity, in line with Problem 1.(ii).

(c) Consider $f, g \in C^\infty(\mathfrak{g}^*)$ and their linear derivatives $df, dg : \mathfrak{g}^* \longrightarrow \mathbb{R}$ at $0 \in \mathfrak{g}^*$ where we identify $T_0\mathfrak{g}^* \cong \mathfrak{g}^*$. Note that we can consider them as elements $df, dg \in \mathfrak{g}$, as $\mathfrak{g}^{**} \cong \mathfrak{g}$. Show that

$$\{\cdot,\cdot\} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \longrightarrow C^\infty(\mathfrak{g}^*),$$
$$\{f, g\}(\xi) := \xi([df, dg]), \quad \forall \xi \in \mathfrak{g},$$

is a Poisson bracket on $C^\infty(\mathfrak{g}^*)$.[6]

(d) A function $f \in C^\infty(\mathfrak{g}^*)$ is called a *Casimir*[7] if $\{f, g\} = 0$ for all $g \in C^\infty(\mathfrak{g}^*)$. Find a Casimir for $\mathfrak{g} = \mathfrak{sl}_n$, $\forall n \in \mathbb{N}$. (Start with $n = 2$.)

(e) (Optional) Find n Casimirs for each of the two orthogonal Lie algebras $\mathfrak{g} = \mathfrak{so}(2n)$ and $\mathfrak{g} = \mathfrak{so}(2n+1)$ of skew-symmetric matrices.

Problem 6. (The Little Arnol'd-Liouville Theorem) Let $F = (f_1, \ldots, f_n) : M \longrightarrow \mathbb{R}^n$ be an integrable system on (M, ω). Consider a regular value $c \in \mathbb{R}^n$ of F, so that $F^{-1}(c) \subseteq M$ is a smooth *compact* submanifold.

(1) Let $p \in F^{-1}(c)$. Prove that $g_i^t(p) \in F^{-1}(c)$, $\forall t \in \mathbb{R}$, $i \in [1, n]$.

(2) Show that the flows g_i^t of the Hamiltonian vector fields X_{f_i} are complete, i.e. the flow exists for all time $t \in \mathbb{R}$.

(3) Let $p \in F^{-1}(c)$. Show that the flows commute:

$$g_i^{t_1}(g_j^{t_2}(p)) = g_j^{t_2}(g_i^{t_1}(p)), \quad t_1, t_2 \in \mathbb{R}, \quad \forall i, j \in [1, n].$$

(4) Let $p \in F^{-1}(c)$. Show that the map

$$\tau : \mathbb{R}^n \longrightarrow F^{-1}(c), \tau(t_1, \ldots, t_n) = (g_n^{t^n} \circ \ldots \circ g_1^{t^1})(p)$$

is a local diffeomorphism. Conclude that $\tau^{-1}(p)$ is a discrete subgroup of $\mathbb{R}^n$.

(5) By using the classification of discrete subgroups of $\mathbb{R}^n$, show that

$$F^{-1}(c) \cong \mathbb{S}^1 \times \overset{(n)}{\cdots} \times \mathbb{S}^1,$$

i.e. the fiber $F^{-1}(c)$ is diffeomorphic to an n-torus.

(6) (Optional) Show that $\omega|_{F^{-1}(c)} \equiv 0$, i.e. the fibers of F are Lagrangian.

Problem 7. (Poisson Brackets and Incompressible Fluids) In 1757, L. Euler modeled the evolution of an inviscid flow, by studying the PDE satisfied by its flow velocity vector $v(x, t)$, x being the position of a fluid particle and $t \in \mathbb{R}_{\geq 0}$ its time evolution.[8] If p is the pressure function and we assume mass density $\rho \equiv 1$, they read:

$$\partial_t v + \nabla_v v = -\nabla p, \quad div(v) = 0.$$

[6] You may assume that $[\cdot,\cdot]$ is a Lie bracket, as stated in (b).

[7] In particular, a Casimir is a constant of motion for *any* Hamiltonian of *any* physical system you ever consider on $\mathfrak{g}^*$.

[8] Published in the "Principes généraux du mouvement des fluides" in the Mémoires de l'Académie des Sciences de Berlin.

These are known as Euler's incompressible fluid equations, where ∇ is the gradient if the fluid is in $(\mathbb{R}^n, g_{\mathrm{st}})$, and in general it is the Levi-Civita connection of (M, g).

Let (M, g) be a Riemannian manifold, and for this problem you are welcome to choose $(M, g) = (\mathbb{R}^n, g_{\mathrm{st}})$, where g_{st} is the standard flat metric, if you prefer. Let us consider the group G of volume preserving diffeomorphisms

$$\mathrm{Diff}^{\mu_g}(M) := \{\varphi \in \mathrm{Diff}(M) : \varphi^*(\mu_g) = \mu_g\}, \quad \mu_g \text{ the volume form of } g.$$

(i) Show that its Lie algebra $\mathfrak{g} = T_{Id}\,\mathrm{Diff}^{\mu_g}(M)$ is isomorphic to the vector space of divergence-free vector fields on M:

$$T_{Id}\,\mathrm{Diff}^{\mu_g}(M) \cong \{u \in \Gamma(TM) : div(v) = 0\}.$$

Hint: First show that $T_{Id}\,\mathrm{Diff}(M)$ is isomorphic to the vector space of vector fields on M. Then show that volume-preserving translates into $div(v) = 0$.

(ii) Let us consider the motion of a fluid particle, which is described by a path

$$\varphi : \mathbb{R} \longrightarrow \mathrm{Diff}(M), \quad t \longmapsto \varphi(\cdot, t),$$

so that a fluid particle at $p \in M$ moves to $\varphi(p, t)$ at time t. Consider the following L^2-inner product:

$$\langle X, Y \rangle := \int_M \langle X(p), Y(p) \rangle_g d\mu_g, \quad X, Y \in \mathfrak{g} = T_{Id}\,\mathrm{Diff}^{\mu_g}(M).$$

In particular, the kinetic energy of φ is

$$L(\varphi, \dot{\varphi}) := \frac{1}{2} \int_M \|\dot{\varphi}\|_g^2 d\mu_g.$$

Show that this inner product is right-invariant:

$$\langle X \circ \vartheta, Y \circ \vartheta \rangle = \langle X, Y \rangle, \quad \forall \vartheta \in \mathrm{Diff}^{\mu_g}(M).$$

(iii) By definition, we generalize to the infinite-dimensional setting by stating that a trajectory $\varphi : [0, 1] \longrightarrow \mathrm{Diff}^{\mu_g}(M)$ is geodesic if it extremizes the length functional

$$l(\varphi) = \int_0^1 \sqrt{\langle \dot{\varphi}, \dot{\varphi} \rangle} dt.$$

Note that we can ask the parametrization to be at constant speed $\langle \dot{\varphi}, \dot{\varphi} \rangle$, as the value of the functional is independent of the parametrization.

Show that φ is a critical point for the Lagrangian given by the kinetic energy above if and only if φ is a geodesic parametrized at constant speed.

(iv) Let φ be a geodesic $\mathrm{Diff}^{\mu_g}(M)$ parametrized at constant speed. Show that $v(t) := \dot{\varphi}(t) \circ \varphi(t)^{-1}$ is a solution to the Euler equations for incompressible fluids

$$\partial_t v + \nabla_v v = -\nabla p, \quad div(v) = 0,$$

for some unique function p, up to additive constants.

Similar arguments lead to many other interesting PDEs. For instance, the same argument applied to the group of diffeomorphisms preserving a contact structure lead to the Camassa–Holm equation, modeling waves in shallow water.