MAT 265: PROBLEM SET 5

DUE TO FRIDAY DEC 11 AT 9:00PM

Abstract. This is the fifth problem set for the graduate course Mathematical Quantum Mechanics in the Fall Quarter 2020. It was posted online on Monday Nov 23 and is due Friday Dec 11 at 9:00am at 9:00pm via online submission.

Purpose: The goal of this assignment is to review and practice the path integral, quantum propagators and the basics of field theory. In particular, we would like to become familiar with examples of computations with the path integral, classical fields (namely Klein-Gordon, Yang-Mills and Chern-Simons) and some of their quantizations.

Task and Grade: Solve two of the six problems below. Each Problem is worth 50 points.The maximum possible grade is 100 points. Despite the task being two problems, I strongly encourage you to work on all the problems.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Textbook: We use "A Brief Introduction to Physics for Mathematicians" by I. Dolgachev, which is freely available in the course website.

Problem 1. (The 1-dimensional Free Particle: Again!) Consider a quantum particular with mass $m \in \mathbb{R}^+$ moving in the real line $\mathbb{R}$. Let $K(q,t;q_0,0)$ be the propagator from (expected) position $q_0 \in \mathbb{R}$ at time $t_0 = 0$ to (expected) position $x \in \mathbb{R}$ at time t. The Lagrangian is thus

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2.$$

In this problem, we allow ourselves to work with path integrals.

(1) Using the canonical quantization for quantum mechanics, show that the propagator is given by

$$K(q,t;x_0,0) = \sqrt{\frac{m}{2\pi i\hbar t}}e^{im(q-q_0)^2/(2t\hbar)}.$$

(2) Now using the path integral, show that the propagator is given by

$$K(q,t;x_0,0) = \sqrt{\frac{m}{2\pi i\hbar t}}e^{im(q-q_0)^2/(2t\hbar)}.$$

(*This is an example where finding the propagator via classical quantum mechanics is simpler than with the path integral. In general, path integrals are usually more efficient, as one does not need to solve the Schrödinger equation.*)

(3) Compute the general propagator $K(q, t; q_0, t_0)$ starting at the point q_0 at time t_0 and ending at q in time t.

(4) Show that the wavefunction $\psi(q, t) = K(q, t; 0, 0)$ satisfies $\psi(q, 0) = \delta(q)$ and interpret this result physically.

(5) Show that the Green function $G(q, t; q_0, 0)$ for the Schrödinger equation[1] is

$$G(q, t; q_0, t_0) = \frac{u_0(t - t_0)}{i\hbar} K(q, t; q_0, t_0).$$

Namely, show that

$$i\hbar \partial_t G(q, t; q_0, t_0) = \left(-\frac{\hbar^2}{2m} \partial_q^2 \cdot G(q, t; q_0, t_0) \right) \cdot \delta(q - q_0)\delta(t - t_0).$$

where u_0 is the Heaviside step function centered at 0.

Problem 2. (Courtesy of A. Waldron) Solve each of the three parts below:

(1) Compute the expected value of the operators $\hat{x}$ and $\hat{p}$ with respect to the coherent state $|z\rangle = e^{za^\dagger}|z\rangle$. Show that this state saturates Heisenberg's uncertainty bound.

(2) The position space propagator for the harmonic oscillator used to compute the probability that a quantum spring evolves from an initial position x_i to a final one x_f in time t is given by

$$K(x_f, x_i; t) = \frac{1}{\sqrt{2\pi i\hbar \sin t}} \exp\Big(- \frac{(x_f - x_i)^2 + (x_f^2 + x_i^2)(\cos \Delta t - 1)}{2i\hbar \sin t} \Big).$$

Use the coherent state propagator result to derive this expression.

(Hint: compute the wavefunction of the state $|z\rangle$ as a function of x.)

(3) Use Feynman diagrams to compute as many terms as you (reasonably) can in the asymptotic series expansion of the integral

$$K(g) := \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2 - \frac{g}{4!}x^4}.$$

Use your result to compute the first few terms in the asymptotic series for $\log K(g)$. Try to express this also as a sum of diagrams and comment on any patterns you happen to observe.

Problem 3. (Electromagnetic Field as $U(1)$-Gauge Theory) In this problem, we derive Maxwell's equations for electromagnetism in the vacuum from the Yang-Mills Lagrangian with gauge group $G = U(1)$. Let $\mathbb{R}^3$ have coordinates (q_1, q_2, q_3), with associated space-time $\mathbb{R}^3 \times \mathbb{R}_t$, and consider the electric field and magnetic fields:

[1] Recall that the Green function of a PDE is the solution obtained when you have a δ source.

$$E(q,t) = E_1 dq_1 + E_2 dq_2 + E_3 dq_3, \qquad B(q,t) = (B_1 dq_2 dq_3 + B_2 dq_3 dq_1 + B_3 dq_1 dq_2),$$

where $E_i, B_i \in C^\infty(\mathbb{R}^3 \times \mathbb{R}_t)$. Note that $E \in \Omega^1(\mathbb{R}^3 \times \mathbb{R}_t)$ is a 1-form and $B \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}_t)$ is a 2-form.[2] Let $* : \Omega^k(\mathbb{R}^3) \longrightarrow \Omega^{3-k}(\mathbb{R}^3)$ and Let $* : \Omega^k(\mathbb{R}^4) \longrightarrow \Omega^{4-k}(\mathbb{R}^4)$ denote the Hodge star operator in $\mathbb{R}^3$ and $\mathbb{R}^3 \times \mathbb{R}$. For instance, in $\mathbb{R}^3$ we will have

$$*dq_1 = dq_2 dq_3, \quad *dq_2 = dq_3 dq_1, \quad *dq_3 = dq_1 dq_2.$$

Maxwell's equations in the vacuum read

$$dB = 0, \quad dE = (\partial_t B), \quad d(*E) = 0, \quad *d(*B) = -\frac{1}{c^2}\partial_t E,$$

where c is the speed of light.[3] The Hodge star operator is applied as forms in $\mathbb{R}^3$, and thus sends 1-forms to 2-forms, and viceversa.

(1) First, show that the equations above are equivalent to the Maxwell equations we teach in vector calculus:

$$\nabla \cdot E = 0, \quad \nabla \cdot B = 0, \quad \nabla \times E = -\partial_t B, \quad \nabla \times B = \frac{1}{c^2}\partial_t E.$$

(2) Let $d^* := *d*$ denote the adjoint of d, and consider the operator $\Delta := dd^* + d^*d$. Show that Δ is the Laplace operator in $(\mathbb{R}^3, g_{\mathrm{st}})$ where g_{st} is the standard flat metric.

(3) Show that the electric and magnetic fields solve the wave equation:

$$\frac{1}{c^2}\partial_t^2 E - \Delta E = 0, \quad \frac{1}{c^2}\partial_t^2 B - \Delta B = 0.$$

(4) Let us consider the *electromagnetic field*[4] in space-time $\mathbb{R}^3 \times \mathbb{R}_t$:

$$F = B + E \wedge dt \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}_t).$$

Show that Maxwell's Equation are equivalent to

$$dF = 0, \quad d(*F) = 0,$$

i.e. F is a *closed* $(dF = 0)$ and *co-closed* $(d(*F) = 0)$ 2-form.

(5) Deduce from Part (4) that Maxwell's Equations are Lorentz invariant. That is, any isometry of Minkowski space $(\mathbb{R}^3 \times \mathbb{R}_t, g_{Mink})$, $g_{Mink} = g_{\mathrm{st}} - dt \otimes dt$, preserves these equations.

(6) Since $F \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}_t)$ is a closed 2-form, there exists a 1-form $A \in \Omega^1(\mathbb{R}^3 \times \mathbb{R}_t)$ such that $dA = F$. Physically, A is known as the *electromagnetic potential*.[5] Show that Maxwell's Equations in terms of A just read $d^*dA = 0$.

[2] In case you might have been taught that E, B where vector-fields, note that we could have dualized by letting (E_1, E_2, E_3) and (B_1, B_2, B_3) be components of a vector field.

[3] Since waves in the vacuum propagate at the speed of light.

[4] You may start to appreciate the versality with which the word *field* is used. Here a field means a 2-form. Two lines above it meant a 1-form or a 2-form, and a Yang-Mills field will mean something else. Welcome to physics.

[5] The 1959 Aharonov-Bohm effect (check it out, it is cool!) shows that the electromagnetic potential is a physically meaningful quantity: it can be measured.

(7) Show that the Euler-Lagrangian equations for the Yang-Mills Lagrangian for $G = U(1)$ are Maxwell's Equations. Namely, Maxwell's Equations are the equations on A for the critical points of the functional

$$L(A) := \int_{\mathbb{R}^3\times\mathbb{R}} F_A \wedge (*F_A) d\mu,$$

are where $F_A = dA$ is the curvature of the electromagnetic potential $A \in \Omega^1(\mathbb{R}^3 \times \mathbb{R}_t)$.

Disclaimer: Technically, A should be a $i\mathbb{R}$-valued 1-form, as the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$ is the imaginary axis. In this problem, we can forget about that, but for a Yang-Mills theory with Lie Group G one should study $\mathfrak{g}$-valued 1-forms. Similarly, the curvature F_A will in general be $dA + [A, A]/2$, where $[A, A]$ just happens to vanish for an Abelian Lie group.

Problem 4. (Klein-Gordon Field Theory) Let us consider a theory in $\mathbb{R}^3 \times \mathbb{R}_t$ with fields being *real scalar functions* $\varphi : \mathbb{R}^3 \times \mathbb{R}_t \longrightarrow \mathbb{R}$, and the Lagrangian being

$$L(\varphi) := \frac{1}{2}\int_{\mathbb{R}^3} \left((\partial_t\varphi)^2 - |\nabla\varphi|^2 - m^2\varphi^2\right) dq_1dq_2dq_3.$$

The classical action in this field theory is thus

$$S(\varphi) := \int_{t_0}^{t_1} L(\varphi)dt = \frac{1}{2}\int_{t_0}^{t_1}\int_{\mathbb{R}^3} \left((\partial_t\varphi)^2 - |\nabla\varphi|^2 - m^2\varphi^2\right) dq_1dq_2dq_3dt.$$

(1) Show that the Euler-Lagrange equation for this action functional is

$$(\partial_t^2 - \nabla^2 + m^2)\varphi = 0.$$

Note that this is an eigenvalue problem for the *wave equation* operator, the so-called d'Alembert operator $\Box = \partial_t^2 - \nabla^2$.

(2) Find the solutions φ of this equation. (E.g. via Fourier transform.)

(3) The Hamiltonian in this classical field theory reads

$$H(\varphi, \pi) = \frac{1}{2}\int_{\mathbb{R}^3} \left(\pi^2 + |\nabla\varphi|^2 + m^2\varphi^2\right) dq_1dq_2dq_3,$$

where $\pi(q, t) = \partial_t\varphi(q, t)$ is the conjugate field. Show that

$$H(\varphi, \pi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_\tau} \alpha(\tau)^*\alpha(\tau)d\tau_1d\tau_2d\tau_3,$$

where we define $\omega(\tau) = (\tau^2 + m)^{1/2}$ and

$$\alpha(\tau) = \int_{\mathbb{R}^3} e^{i\tau\cdot q}(\omega(\tau)\varphi(q, t) - i\pi(q, t))dq_1dq_2d_3,$$

$$\alpha(\tau)^* = \int_{\mathbb{R}^3} e^{-i\tau\cdot q}(\omega(\tau)\varphi(q, t) + i\pi(q, t))dq_1dq_2d_3.$$

(4) The Hamiltonian H above gets quantized, thanks to Part (3), to the operator

$$\widehat{H} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_\tau} a(\tau)^* a(\tau) d\tau_1 d\tau_2 d\tau_3,$$

where the two operators $a(\tau), a(\tau)^*$ quantize $\alpha(\tau), \alpha(\tau)^*$, that is:

$$a(\tau) = \int_{\mathbb{R}^3} e^{i\tau\cdot q}\left(\omega(\tau)\varphi(q,t) + \frac{\delta}{\delta\varphi(q,t)}\right) dq_1 dq_2 d_3,$$
$$a(\tau)^* = \int_{\mathbb{R}^3} e^{-i\tau\cdot q}\left(\omega(\tau)\varphi(q,t) - \frac{\delta}{\delta\varphi(q,t)}\right) dq_1 dq_2 d_3.$$

(5) Show that the quantum wavefunctional Ψ_0, i.e. a quantum field, given by

$$\Psi_0(\varphi) := \exp\left(-\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3_\tau}\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_q} e^{i\tau(q-y)}\phi(q,t)\phi(y,t)\omega(\tau)\right)$$

is a ground stated for the Klein-Gordon quantum field theory.

(6) (Optional) Find the vacuum to vacuum propagator for the Klein-Gordon quantum field theory.

Problem 5. (Non-Abelian Yang-Mills) Let G be a Lie group. Consider the following theory: fields are connections $A \in \Omega^1(P;\mathfrak{g})$ on a G-principal bundle $\pi : P \longrightarrow \mathbb{R}^3 \times \mathbb{R}_t$ and the Lagrangian is the Yang-Mills functional

$$L(A) := \int_{\mathbb{R}^3\times\mathbb{R}_t} tr(F_A \wedge (*F_A))d\mu.$$

Since $\mathbb{R}^3\times\mathbb{R}_t$ is topologically trivial, any G-bundle is just the product $P = (\mathbb{R}^3\times\mathbb{R}_t)\times G$ and π is the projection onto the first factor $\mathbb{R}^3 \times \mathbb{R}_t$.

(1) Locally, any connection A can be written as a $\mathfrak{g}$-valued 1-form

$$A(q,t) = A_1(q,t)dq_1 + A_2(q,t)dq_2 + A_3(q,t)dq_3 + A_4(q,t)dt, \qquad A_i(q,t) \in \mathfrak{g}.$$

Then the associated covariant derivative, whose integral yields parallel transport, is $D_A := d + A$ and the curvature reads $F_A := D_A^2$. Show that

$$F_A = dA + \frac{1}{2}[A, A],$$

where $[\cdot,\cdot]$ is the Lie bracket of $\mathfrak{g}$ extended to $\mathfrak{g}$-valued 1-forms via the formula: $[A_i dx_i, B_j dx_j] := [A_i, B_j]dx_i dx_j$.[6]

(2) Show that the critical points of the Yang-Mills Lagrangians $L(A)$ are cut out by the Euler-Lagrange equations

$$D_A F_A = 0, \qquad D_A(*F_A) = 0.$$

Solutions to these equations are known as *Yang-Mills connections* and also as *instantons*.[7] There are many of them: they form an ∞-dimensional space $\mathcal{M}$.

[6]Namely, you apply the bracket to the $\mathfrak{g}$-coefficients of the form, and wedge the form parts as usual.
[7]The notation "instanton" comes from a basic Morse theory fact, feel free to ask me about that.

(3) Show that every connection A automatically satisfies $D_A F_A = 0$.
(This is known as the *Bianchi identity* in differential geometry.)

(4) The symmetry group of the principal bundle is given by any section

$$\tilde{\phi} : \mathbb{R}^3 \times \mathbb{R}_t \longrightarrow P,$$

which in this case is tantamount to a function $\phi : \mathbb{R}^3 \times \mathbb{R}_t \longrightarrow G$. The symmetry is given by multiplying the fiber G over $p \in \mathbb{R}^3 \times \mathbb{R}_t$ by the element of the Lie group $\phi(p) \in G$. These symmetries are called *gauge transformations*, and often denoted by g.[8] Show that a gauge transformation acts on the covariant derivative $D_A = d + A$ as

$$D_A \longmapsto g(D_A)g^{-1} = -dg \cdot g^{-1} + gAg^{-1}.$$

(5) Suppose that A is a Yang-Mills instanton and g is a gauge transformation. Show that $g(A)$ is also a Yang-Mills instanton.

Notice that the space of gauge transformations $\mathcal{G}$ is also infinite dimensional. In Yang-Mills gauge theory, the space of Yang-Mills instantons $\mathcal{M}$ is infinite-dimensional, and the space of it symmetries $\mathcal{G}$ is also infinite-dimensional. The kicker is that the quotient $\mathcal{M}/\mathcal{G}$ may be finite-dimensional. (E.g. it will be so for anti-self-dual instantons.)

(6) Suppose that a connection A satisfies $F_A = (*F_A)$. Show that A is a Yang-Mills instanton. Connections that satisfy $F_A = (*F_A)$ are known as *self-dual Yang-mills connections.*

(7) (Optional)[9] Let us consider $G = SU(2)$ and $c_2(P)$ the second Chern class of P. Then the space of self-dual Yang-mills connections modulo gauge transformations has (virtual) dimension $8c_2(P) - 3$.

Problem 6. (Chern-Simons Field Theory) Let M be a 3-manifold, e.g. $M = \mathbb{R}^3$, $k \in \mathbb{N}$, G a Lie group and P a G-principal bundle. The space of Chern-Simons fields is the space of connections on P. The Chern-Simons action on a field A is the quantity $S(A) \in \mathbb{R}/(2\pi\mathbb{Z})$ is given by

$$S(A) := -\frac{k}{2m\pi} \int_M tr\left(AdA + \frac{2}{3}A^3\right) d\mu.$$

(1) Show that the Chern-Simons functional is invariant under gauge transformations. (So S descends to the space of connections modulo the gauge group.)

(2) Prove that the critical points of S are given by flat connections, i.e. connections A with zero curvature $F_A = 0$.[10]

[8] This notation is meant to remind you that you are multiplying by an element $g \in G$ at each fiber.
[9] This requires applying the Atiyah-Singer Index Theorem.
[10] Studying moduli spaces of flat connections in 3-manifold is really good business: it connects with Teichmüller theory, representation theory and all sorts of differential geometries.

(3) Let $K \subseteq M$ be a knot and ρ a representation of G. There is a natural observable associated to the knot:

$$W_K(A) = tr_\rho\left(e^{\int_K A}\right).$$

This observable, which inputs a Chern-Simons field A and outputs a number $W_K(A)$, is known as a *Wilson loop*. Show that $W_K(A)$ is an invariant of the isotopy class of the knot if A is a critical point for Chern-Simons.

(4) The partition function, i.e. the transition amplitude from vacuum at $t = -\infty$ to vacuum at $t = \infty$, in the path integral quantization is given by

$$Z_{cs}(A) = \int e^{iS(A)} d\mu_A.$$

Consider a knot $K \subseteq M = \mathbb{S}^3$, ρ the fundamental representation of $G = SU(2)$ and $K_0 \subseteq \mathbb{S}^3$ the unknot. Then we can consider the modified partition function

$$Z_{cs}(A; K) = \int W_K(A) e^{iS(A)} d\mu_A.$$

Show that $Z_{cs}(A; K)$ divided by $Z_{cs}(A; K_0)$ is the Jones polynomial of K.

(*Hint*: Show that it satisfies the Jones polynomial Skein relations. The variable t in the Jones polynomial will be $t = e^{\frac{2\pi i}{2+k}}$).

This is the start of a fruitful interaction between physics and 3-dimensional topology. E.g. you can change the Lie group $G = SU(2)$, or the representation, to another one, such as $G = SU(N)$ and its infinitely many irreducible (finite-dimensional) representations: then you get knot invariants known as the (colored) HOMFLY-PT polynomials. Even better, in his paper "Khovanov Homology and Gauge Theory" (2012) E. Witten sketched how to obtain Khovanov homology from a 4-dimensional (super-symmetric) Yang-Mills theory.)

(5) (Optional) Read E. Witten's article "Quantum Field Theory and the Jones Polynomial" (1989).