Augmentations, Fillings, and Clusters

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Legendrian Links and Exact Lagrangian Fillings

Definition

Equip $\mathbb{R}^3_{xyz}$ with the standard contact 1-form $\alpha = dz - ydx$.

Definition

A *Legendrian link* is an embedded closed 1-dimensional submanifold $\Lambda \subset \mathbb{R}^3_{xyz}$ such that $\alpha|_\Lambda = 0$.

There are two useful projections when studying Legendrian links.

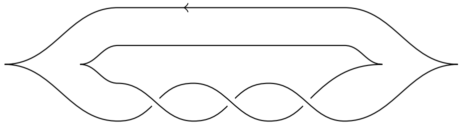
- Front projection $\pi_F : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xz}$; y can be recovered by $y = \frac{dz}{dx}$.
- Lagrangian projection $\pi_L : \mathbb{R}^3_{xyz} \to \mathbb{R}^2_{xy}$; z can be recovered by $z = \int ydx$.

Example

A Legendrian unknot.

Example

For any positive braid $\beta \in \mathrm{Br}_n^+$, its rainbow closure Λ_β is naturally a Legendrian link.



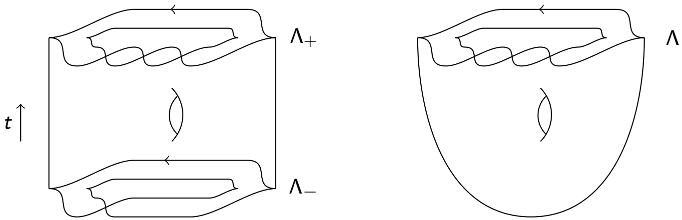
A contact manifold can be viewed as the boundary of a symplectic manifold. The symplectic manifold $\mathbb{R}^4_{xyzt}$ with symplectic form $\omega = d\left(e^t \alpha\right)$ is a *symplectization* of the standard contact $\mathbb{R}^3_{xyz}$.

Definition

Let $\Lambda_\pm$ be two Legendrian links. An *exact Lagrangian cobordism* $L : \Lambda_- \to \Lambda_+$ is a Lagrangian surface $L \subset \mathbb{R}^4_{xyzt}$ such that

- L is asymptotically $\Lambda_\pm$ at $\mathbb{R}^3_{xyz} \times (\pm\infty)$;
- the 1-form $e^t \alpha|_L = df$ for some function f on L that are asymptotically constant.

An exact Lagrangian cobordism $L : \emptyset \to \Lambda$ is also called an *exact Lagrangian filling*.

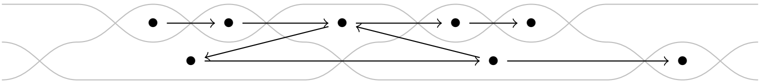


Distinguishing Exact Lagrangian Fillings

- By a result of Chantraine [Cha10], all exact Lagrangian fillings are topologically the same.
- We would like to distinguish exact Lagrangian fillings up to Hamiltonian isotopy.
- Another question: does there exist a Legendrian link with infinitely many non-Hamiltonian isotopic fillings?
- In 2020, different groups came up with examples with infintiely many fillings, answering the last question:
  - R. Casals and H. Gao [CG20] (January): any torus (n, m)-link except $(2, m)$, $(3, 3)$, $(3, 4)$ and $(3, 5)$.
  - R. Casals and E. Zaslow [CZ20] (July): rainbow closures of an infinite family of 3-strand braids.
  - H. Gao, L. Shen and W. [GSW20] (September): rainbow closures of positive braids β other than those are associated with quivers of finite type.
  - R. Casals and L. Ng (upcoming): certain family of Legendrian links including some (-1)-closures of positive braids that are not necessarily rainbow closures of positive braids.
- Among the four projects above, the first two are based on the theory of microlocal sheaves [STZ17, STWZ19], and the latter two are based on symplectic field theory [EHK16]. Surprisingly (or not surprisingly), all of the four projects have direct or indirect connection to cluster theory.

Quiver from Positive Braids

Consider $\Lambda_{(2,1,1,1,2,1,1,1,2,2)} \in \mathrm{Br}_3^+$.

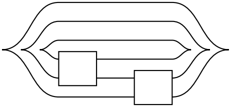


Below are what we call *standard ADE links*.

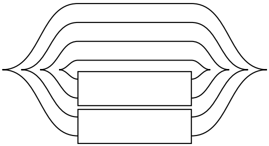
Br_2^+	Br_3^+			
A_r	D_r	E_6	E_7	E_8
s_1^{r+1}	$s_1^{r-2}s_2s_1^2s_2$	$s_1^3s_2s_1^3s_2$	$s_1^4s_2s_1^3s_2$	$s_1^5s_2s_1^3s_2$

Theorem (Gao-Shen-W)

If the rainbow closure of a positive braid is not Legendrian isotopic to a split union of unknots and connect sums of standard ADE *links, then it admits infinitely many non-Hamiltonian isotopic exact Lagrangian fillings.*



connect sum



split union

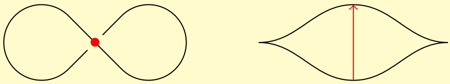
Chekanov-Eliashberg DGA and Augmentation Variety

Definition

Double points in a Lagrangian projection $\pi_L(\Lambda)$ are called *Reeb chords*. In other words, each Reeb chord corresponds to a pair of points in Λ with the same x- and y-coordinates.

Example

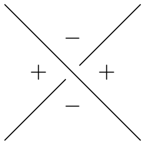
Below are depictions of the unique Reeb chord of the Legendrian unknot example in its Lagrangian projection and front projection.



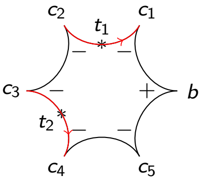
Definition of the Chekanov-Eliashberg dga

We follow [Che02, ENS02] for the construction of the CE dga. Let Λ be an **oriented** Legendrian link **each of whose components has rotation number 0**. We decorate $\pi_L(\Lambda)$ with marked points t_i away from crossings, such that each link component of Λ contains at least (a lift of) one such marked point.

- As an algebra, the CE dga $\mathcal{A}(\Lambda)$ is a $\mathbb{Z}_2$-algebra freely generated by the Reeb chords in $\pi_L(\Lambda)$ and formal variables $t_i^{\pm 1}$ associated with the marked points.
- The degrees of the formal variables $t_i^{\pm 1}$ are 0; the degrees of the Reeb chords are give by the Maslov potential.
- The differential is given by a counting of pseudo-holomorphic disks in $\pi_L(\Lambda)$.



Reeb sign

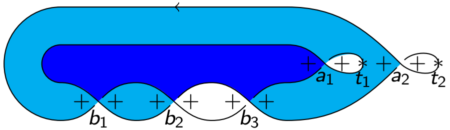


$$w(u) = c_1 t_1^{-1} c_2 c_3 t_2 c_4 c_5$$

$$\partial b = \sum_{c_1,\dots,c_n} \sum_{u \in \mathcal{M}(b; c_1,\dots,c_n)} w(u).$$

Example

Consider the following Legendrian trefoil Λ.



- As an algebra, $\mathcal{A}(\Lambda)$ is generated freely over $\mathbb{Z}_2$ by $t_i^{\pm 1}$, b_i, and a_i.
- The grading on the generators are

$$\left|t_i^{\pm 1}\right| = |b_i| = 0, \qquad |a_i| = 1.$$

- Note that $\mathcal{A}(\Lambda)$ is concentrated in non-negative degrees, so automatically

$$\partial b_i = \partial t_i^{\pm 1} = 0.$$

For ∂a_1 and ∂a_2:

$$\partial a_1 = t_1^{-1} + b_1 + b_3 + b_1 b_2 b_3,$$

$$\partial a_2 = t_2^{-1} + b_2 + t_1 + b_2 b_3 t_1 + t_1 b_1 b_2 + b_2 b_3 t_1 b_1 b_2.$$

Augmentation Variety

Definition

Let Λ be a Legendrian link and let $\mathcal{A}(\Lambda)$ be its CE-dga. Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. An *augmentation* is a dga homomorphism $\epsilon : \mathcal{A}(\Lambda) \to \mathbb{F}$ where $\mathbb{F}$ is concentrated at degree 0 and equipped with the trivial differential. The *augmentation variety* $\mathrm{Aug}(\Lambda)$ is defined to be the moduli space of augmentations of $\mathcal{A}(\Lambda)$.

- Note that an augmentation ϵ basically assigns $\mathbb{F}$-values to the degree 0 generators of $\mathcal{A}(\Lambda)$ subject to the conditions $\partial a = 0$ for degree 1 Reeb chords a. Therefore augmentation varieties are always affine varieties.
- If $\mathcal{A}(\Lambda)$ is concentrated in non-negative degrees, then

$$\mathrm{Aug}(\Lambda) \cong \mathrm{Spec}\ H_0\left(\mathcal{A}^c(\Lambda)\right),$$

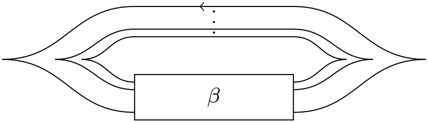
where ^{c} stands for the commutatization of $\mathcal{A}(\Lambda)$.

Example

In our last example, we have $\partial a_1 = t_1^{-1} + b_1 + b_3 + b_1 b_2 b_3$ and $\partial a_2 = t_2^{-1} + b_2 + b_1 + b_2 b_3 t_1 + t_1 b_1 b_2 + b_2 b_3 t_1 b_1 b_2$. Setting these two equations to 0 cuts out a 3-dimensional affine variety in $\mathbb{F}^3_{b_1,b_2,b_3} \times \left(\mathbb{F}^\times\right)^2_{t_1,t_2}$.

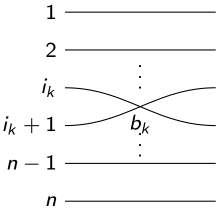
Augmentation Varieties of Positive Braid Closures

For a positive braid $\beta \in \mathrm{Br}_n^+$, Λ_β has $n+l$ number of Reeb chords in total. There are l Reeb chords coming from the crossings of β, which we denote by b_k $(1 \le k \le l)$, and n Reeb chords coming from the right cusps, which we denote by a_i. Their gradings are $|b_k| = 0$ and $|a_i| = 1$.



For the crossing i_k between the i_kth and i_k+1st strands, we associate an $n \times n$ matrix $Z_{i_k}(b_k)$ of the form

$$Z_{i_k}(b_k) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & b_k & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$



Let $\beta = (i_1, \dots, i_l)$ be a positive braid in Br_n^+. Define $M := Z_{i_1}(b_1) Z_{i_2}(b_2) \cdots Z_{i_l}(b_l)$.

Theorem (Gao-Shen-W)

The homology $H_0(\mathcal{A}(\Lambda_\beta))$ is generated by $b_1, \cdots, b_l, t_1^{\pm 1}, \cdots, t_n^{\pm 1}$, modulo the relations $t_k^{-1} = M_k$ for $1 \leq k \leq n$, where M_k is the Gelfand-Retakh quasi-determinant [GR91] of the upper-left $k \times k$ submatrix of M with respect to the (k,k)-entry.

Corollary (Gao-Shen-W.)

Let M^c be the matrix obtained by abelianizing entries of the matrix M. Then the augmentation variety $\mathrm{Aug}(\Lambda_\beta)$ is isomorphic to the non-vanishing locus of $\prod_{m=1}^n \Delta_m(M^c)$ in $\mathbb{F}^l_{b_1,\dots,b_l}$, where Δ_m denotes the determinant of the $m \times m$ submatrix at the upper left corner.

We recognize that the non-vanishing locus stated above is isomorphic to a double Bott-Samelson cell, which is known to be a cluster variety.

Double Bott-Samelson Cells and Cluster Varieties

Double Bott-Samelson Cells

Double Bott-Samelson cells were introduced in a joint work with L. Shen [SW19]. Let G be a Kac-Peterson group with a pair of opposite Borel subgroups $B_\pm$. Denote $xB_\pm \xrightarrow{w} yB_\pm$ if $x^{-1}y \in B_\pm w B_\pm$ and denote $xB_- \overset{w}{\text{———}} yB_+$ if $x^{-1}y \in B_- w B_+$.

$$P_{i_1} \underset{B_+}{\times} P_{i_2} \underset{B_+}{\times} \cdots \underset{B_+}{\times} P_{i_l} = \bigsqcup_{\mathrm{j} \preceq \mathrm{i}} (B_+ s_{j_1} B_+) \underset{B_+}{\times} (B_+ s_{j_2} B_+) \underset{B_+}{\times} \cdots \underset{B_+}{\times} (B_+ s_{j_m} B_+)$$

$[x_1, \ldots, x_m] \in (B_+ s_{j_1} B_+) \underset{B_+}{\times} \cdots \underset{B_+}{\times} (B_+ s_{j_m} B_+)$ gives rise to a unique sequence

$$B_+ \xrightarrow{s_{j_1}} x_1 B_+ \xrightarrow{s_{j_2}} x_1 x_2 B_+ \xrightarrow{s_{j_3}} \cdots \xrightarrow{s_{j_m}} x_1 \cdots x_m B_+.$$

Denote left cosets of B_+ as B^i and denote left cosets of B_- as B_j. Let $U_\pm := [B_\pm, B_\pm]$. Denote left cosets of U_+ as A^i and denote left cosets of U_- as A_j.

Definition (Shen-W.)

Let (β, γ) be a pair of positive braids associated to G and let $\mathrm{i} = (i_1, \ldots, i_l)$ and $\mathrm{j} = (j_1, \ldots, j_m)$ be words of β and γ. The *decorated double Bott-Samelson cell* is

$$\mathrm{Conf}_\beta^\gamma(\mathcal{C}) = \left\{ \begin{array}{ccccccc} A_0 & \xrightarrow{s_{j_1}} & B_1 & \xrightarrow{s_{j_2}} & \cdots & \xrightarrow{s_{j_m}} & B_m \\ | & & & & & & | \\ B^0 & \xrightarrow[s_{i_1}]{} & B^1 & \xrightarrow[s_{i_2}]{} & \cdots & \xrightarrow[s_{i_l}]{} & B^l \end{array} \right\} \Big/ G.$$

Decorated Double Bott-Samelson Cells

For a point in $\mathrm{Conf}^e_\beta(\mathcal{C})$, we can use the G-action to move the configuration such that the left vertical edge is $U_- \;\rule[0.5ex]{2em}{0.4pt}\; B_+$. There are $\mathbb{A}^1$-many flags that are s_{i_1} away from B_+, and they can be parametrized as $z_{i_1}(q_1)B_+$, where $z_i(q) = \varphi_i \begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix}$.

$$
\begin{tikzcd}
 & & & U_- \arrow[dlll, dash] \arrow[drr, dash] & & \\
B_+ \arrow[r, "s_{i_1}"'] & z_{i_1}(q_1)B_+ \arrow[r, "s_{i_2}"'] & z_{i_1}(q_1)z_{i_2}(q_2)B_+ \arrow[r, "s_{i_3}"'] & \cdots \arrow[r, "s_{i_l}"'] & & z_{i_1}(q_1)\cdots z_{i_l}(q_l)B_+
\end{tikzcd}
$$

In order for $U_- \;\rule[0.5ex]{5em}{0.4pt}\; z_{i_1}(q_1)\cdots z_{i_l}(q_l)B_+$ to hold, we need all principal minors of $z_{i_1}(q_1)\cdots z_{i_l}(q_l)$ to be non-zero.

Theorem (Gao-Shen-W.)

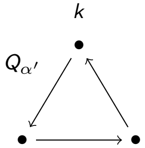
For $G = SL_n$, the map $\gamma^*(q_i) = b_i$ defines a canonical isomorphism $\gamma : \mathrm{Aug}\left(\Lambda_\beta\right) \xrightarrow{\cong} \mathrm{Conf}^e_\beta(\mathcal{C})$ (over characteristic 2). Moreover, this canonical isomorphism does not depend on the choice of word i for β.

We proved in [SW19] that double Bott-Samelson cells, both decorated and undecorated, are cluster varieties. By a pull-back through γ, we obtain a cluster variety structure on $\mathrm{Aug}\left(\Lambda_\beta\right)$.

Crash Course on Cluster Varieties

- Cluster theory began with the invention of cluster algebras by Fomin and Zelevinsky [FZ02]. Fock and Goncharov introduced cluster varieties [FG09] as geometric counterparts of cluster algebras.
- There are two types of cluster varieties, one is called K2 or $\mathcal{A}$, and the other one is called Poisson or $\mathcal{X}$. We will focus on the $\mathcal{A}$ type today.
- A cluster variety is an affine variety together with an atlas (up to codimension 2) of open torus charts called *cluster charts*.
- Each torus chart α is equip with a set of *cluster coordinates* $(A_{i;\alpha})$ and a quiver Q_α. The cluster coordinates are indexed by the vertex set of the quiver Q_α.
- Charts are glued via a process called *cluster mutation*. For any (unfrozen) vertex k of Q_α, the cluster mutation μ_k relates α to another cluster chart α', and their cluster coordinates are related by

$$A_{i;\alpha'} = \begin{cases} \dfrac{1}{A_{k;\alpha}} \left(\displaystyle\prod_{\substack{j \to k \\ \text{in } Q_\alpha}} A_{j;\alpha} + \prod_{\substack{k \to j \\ \text{in } Q_\alpha}} A_{j;\alpha} \right) & \text{if } i = k, \\ A_{i;\alpha} & \text{otherwise.} \end{cases}$$



On the other hand, their quivers $Q_{\alpha'}$ and Q_α are related by a *quiver mutation* at the quiver vertex k.

Admissible Fillings and Cluster Charts

Admissible Fillings

- Ekholm, Honda, and Kálmán [EHK16] constructed a contravariant functor Φ^* from the cobordism category of Legendrian links in the symplectization $\mathbb{R}^4_{xyzt}$ to the category of dga's, mapping Λ to $\mathcal{A}(\Lambda)$.
- They further proved that if two exact Lagrangian cobordisms $L, L' : \Lambda_- \to \Lambda_+$ are Hamiltonian isotopic, then the induced dga homomorphisms Φ_L^* and $\Phi_{L'}^*$ are chain-homotopic, which then implies that they induce equal homomorphisms between the 0th homologies

$$\Phi_L^* = \Phi_{L'}^* : H_0\left(\mathcal{A}\left(\Lambda_+\right)\right) \to H_0\left(\mathcal{A}\left(\Lambda_-\right)\right).$$

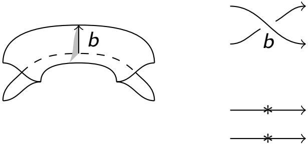
- Consequently, if the dga $\mathcal{A}\left(\Lambda_\pm\right)$ are concentrated in non-negative degrees, then Hamiltonian isotopic exact Lagrangian cobordisms induce an equal augmentation variety morphisms

$$\Phi_L = \Phi_{L'} : \mathrm{Aug}\left(\Lambda_-\right) \to \mathrm{Aug}\left(\Lambda_+\right).$$

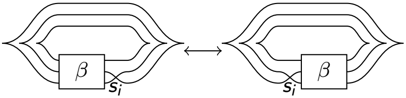
- In this project, we focus on a special family of exact Lagrangian fillings which we call *admissible fillings*. They are compositions of the following:
  - saddle cobordisms;
  - cyclic rotations;
  - braid moves;
  - filling an unknot.

  We choose these as building blocks because we can compute their functorial morphisms explicitly by based on methods developed by Eckholm-Honda-Kálmán [EHK16] and Sivek [Siv11].

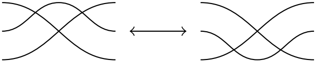
Fillings and Clusters



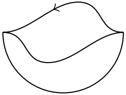
saddle cobordism



cyclic rotation



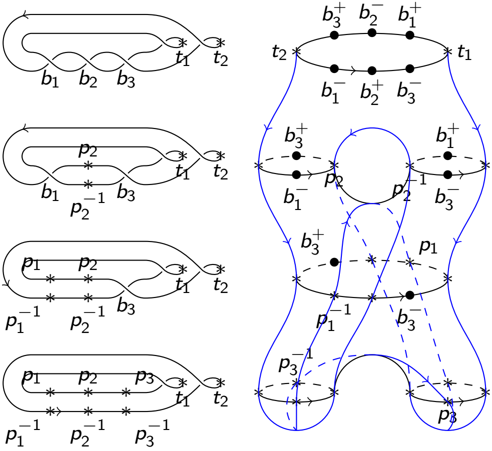
braid move



filling an unknot

Theorem (Gao-Shen-W.)

If L is an admissible filling of Λ_β, then $\Phi_L : \mathrm{Aug}(\emptyset, \mathcal{P}) \to \mathrm{Aug}\left(\Lambda_\beta\right)$ is an open embedding of an algebraic torus, and the image of Φ_L coincides with a cluster chart of $\mathrm{Aug}\left(\Lambda_\beta\right)$. Moreover, if two admissible fillings L and L' correspond to two distinct cluster seeds, then L and L' are not Hamiltonian isotopic.



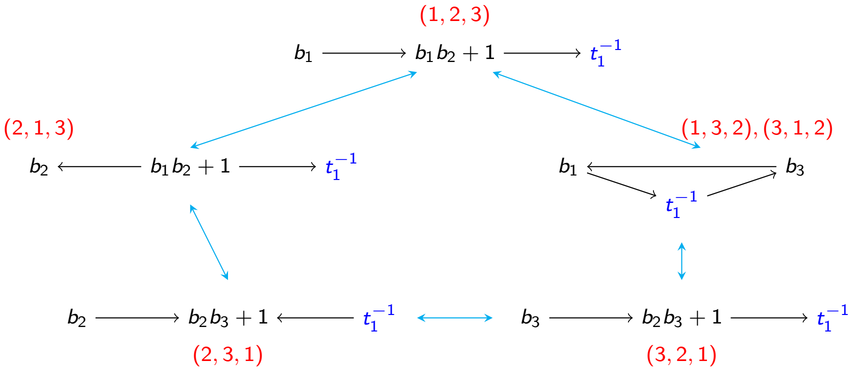
$$
\begin{array}{ccc}
b_1 & b_2 & b_3 \\
\downarrow & \downarrow & \downarrow \\
b_1 + p_2^{-1} & p_2 & b_3 + p_2^{-1} \\
\downarrow & \downarrow & \downarrow \\
p_1 + p_2^{-1} & p_2 & b_3 + p_2^{-1}p_1^{-1}p_2^{-1} + p_2^{-1} \\
\downarrow & \downarrow & \downarrow \\
p_1 + p_2^{-1} & p_2 & p_3 + p_2^{-1}p_1^{-1}p_2^{-1} + p_2^{-1}
\end{array}
$$

$$p_1 p_2 p_3 t_1 = 1, \qquad p_1^{-1} p_2^{-1} p_3^{-1} t_2 = 1.$$

$$\mathrm{Aug}\left(\emptyset, \mathcal{P}\right) \cong \left(\mathbb{F}^\times\right)^3_{p_1,p_2,p_3} \longrightarrow \mathrm{Aug}\left(\Lambda_{(1,1,1)}\right) \subset \mathbb{F}^3_{b_1,b_2,b_3}.$$

Example Continued

The Legendrian trefoil augmentation variety $\mathrm{Aug}\left(\Lambda_{(1,1,1)}\right)$ is cut out from $\mathbb{F}^3_{b_1,b_2,b_3} \times \mathbb{F}^\times_{t_1}$ by the equation $t_1^{-1} + b_1 + b_3 + b_1 b_2 b_3 = 0$. It has five clusters.



We have a calculator to compute Φ_L and the corresponding cluster for any admissible filling L of the rainbow closure of a positive braid.

Full Cyclic Rotation and Cluster DT Transformation

Definition

We define the *full cyclic rotation* $R : \Lambda_\beta \to \Lambda_\beta$ to be the exact Lagrangian cobordism that rotates the crossings of β one-by-one from right to left.

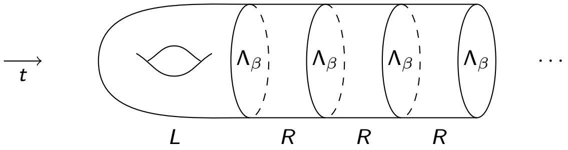
Theorem (Gao-Shen-W.)

The functorial isomorphism $\Phi_R : \mathrm{Aug}\left(\Lambda_\beta\right) \to \mathrm{Aug}\left(\Lambda_\beta\right)$ *equals* DT^2*, where* DT *is the cluster Donaldson-Thomas transformation on* $\mathrm{Aug}\left(\Lambda_\beta\right)$*.*

The cluster DT transformation is a biregular automorphism on a cluster variety and it permutes the cluster charts on a cluster variety. It is known that if Q is an acyclic quiver of infinite type, then this permutation is of infinite order [LLMSS20].

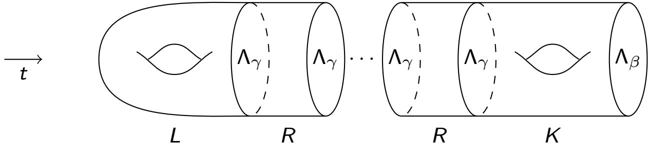
Corollary (Gao-Shen-W.)

Suppose $\mathrm{Aug}\left(\Lambda_\beta\right)$ *has a quiver that is acyclic and of infinite type. Then for any admissible filling* L *of* Λ_β*, the set* $\left\{R^k \circ L \;\middle|\; k \geq 0\right\}$ *is an infinite family of non-Hamiltonian isotopic exact Lagrangian fillings.*



Most Positive Braid Closures Admit Infinitely Many Fillings

- For a general positive braid β, we employ an exhaustive argument in [GSW20] to show that if Λ_β is not Legendrian isotopic to a split union of unknots and connect sums of standard ADE links, then there exists a positive braid γ with an acyclic quiver of infinite type and with an admissible cobordism $K : \Lambda_\gamma \to \Lambda_\beta$.



- We also prove that the functorial morphism $\Phi_K : \mathrm{Aug}(\Lambda_\gamma) \to \mathrm{Aug}\left(\Lambda_\beta\right)$ is an open embedding that preserves the cluster structure.
- Therefore, by composing K with the infinitely many non-Hamiltonian isotopic admissible fillings of Λ_γ we get infinitely many non-Hamiltonian isotopic admissible fillings of the form $K \circ R^k \circ L$ for Λ_β as well.

Preview on Upcoming Results

- There is a characteristic 0 version of CE dga's and augmentation varieties. When Λ_β is decorated with only one marked point per link component, the characteric 0 augmentation variety $\mathrm{Aug}\left(\Lambda_\beta\right)$ is a symplectic variety.
- There is a Deohdar stratification on double Bott-Samelson cells [SW19], and there is a stratification on augmentation variety by normal rulings [HR14]. These two stratifications coincide under the canonical isomorphism γ.
- For any Legendrian link Λ one can also define an *augmentation stack* $\mathrm{Aug}_{st}(\Lambda)$ [NRSSZ15], and the augmentation stack possesses an unfrozen cluster $\mathcal{X}$ structure [STWZ19, SW19]. For any positive braid β, the projection map $\mathrm{Aug}\left(\Lambda_\beta\right) \to \mathrm{Aug}_{st}\left(\Lambda_\beta\right)$ coincides with the cluster theoretical map $p : \mathcal{A} \to \mathcal{X}^{\mathrm{uf}}$.

Thank You!

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