Parabolic Hilbert schemes and representation theory

José Simental (UC Davis)
(joint work with Eugene Gorsky and Monica Vazirani)

UC Davis Algebraic Geometry Seminar
October 14, 2020

Parabolic Hilbert schemes and representation theory

José Simental (UC Davis)
(joint work with Eugene Gorsky and Monica Vazirani)

UC Davis Algebraic Geometry Seminar
October 14, 2020

I. The spaces

- Let $C = \{f(x,y) = 0\} \subseteq \mathbb{C}^2$ be a plane curve. We will assume $0 \in C$.

I. The spaces

- Let $C = \{f(x, y) = 0\} \subseteq \mathbb{C}^2$ be a plane curve. We will assume $0 \in C$.

- The Hilbert scheme $\mathrm{Hilb}^k(C)$ is the moduli space of codimension k ideals inside the algebra of functions, $\mathcal{O}_C = \mathbb{C}[x, y]/(f)$.
$$\mathrm{Hilb}^k(C) \coloneqq \{I \subseteq \mathcal{O}_C \,|\, \dim(\mathcal{O}_C/I) = k\}.$$

I. The spaces

- Let $C = \{f(x,y) = 0\} \subseteq \mathbb{C}^2$ be a plane curve. We will assume $0 \in C$.

- The Hilbert scheme $\mathrm{Hilb}^k(C)$ is the moduli space of codimension k ideals inside the algebra of functions, $\mathcal{O}_C = \mathbb{C}[x,y]/(f)$.

$$\mathrm{Hilb}^k(C) \coloneqq \{I \underset{\text{ideal}}{\subseteq} \mathcal{O}_C \mid \dim(\mathcal{O}_C/I) = k\}.$$

- The punctual Hilbert scheme $\mathrm{Hilb}^k(C,0)$ is the moduli space of codimension k ideals inside the algebra of germs of functions around 0, $\mathcal{O}_C^{\wedge 0} = \mathbb{C}[\![x,y]\!]/(f)$.

$$\mathrm{Hilb}^k(C,0) \coloneqq \left\{I \underset{\text{ideals}}{\subseteq} \mathcal{O}_C^{\wedge 0} \middle| \dim(\mathcal{O}_C^{\wedge 0}/I) = k\right\}.$$

Example. If $0 \in C$ is a smooth point, then $\mathcal{O}_C^{\wedge 0} = \mathbb{C}[[t]]$, so
$$\mathrm{Hilb}^k(C,0) = \{\mathrm{pt}\} \text{ for every } k.$$

"(t^k)"

Example. If $0 \in C$ is a smooth point, then $\mathcal{O}_C^{\wedge 0} = \mathbb{C}[\![t]\!]$, so

$$\mathrm{Hilb}^k(C,0) = \{\mathrm{pt}\} \text{ for every } k.$$

Example. Let $C = \{xy = 0\}$. Then $\mathrm{Hilb}^k(C,0) = \{\mathrm{pt}\}$ for $k = 0, 1$.

$(\alpha,\beta) \neq (0,0)$

- $\mathrm{Hilb}^2(C,0) = \{(x^2, y^2, \alpha x + \beta y)\} = \mathbb{P}^1$.

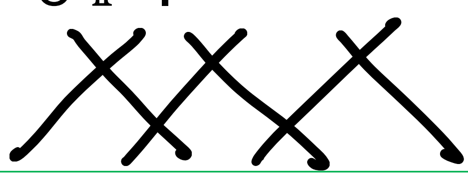
$\mathbb{P}^1$ $\mathbb{P}^1$

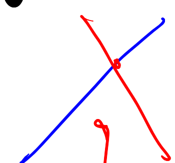
$(\alpha,\beta) \neq (0,0)$

- $\mathrm{Hilb}^3(C,0) = \{(x^2,y^2),(x,y^3),(x^3,y),(\alpha x + \beta y^2),(\alpha x^2 + \beta y)\}$
$= \mathbb{P}^1 \cup \mathbb{P}^1.$

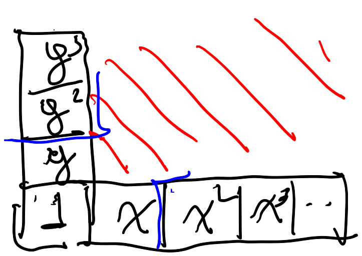
$k-1$ $\mathbb{P}^1$'s $(x^2, y^3, \alpha x + \beta y^2)$

- $\mathrm{Hilb}^k(C,0) = \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1.$





(x^2, y^2)



From now on, we fix coprime integers n and m, and we consider the curve

$$C = \{x^m = y^n\}.$$

From now on, we fix coprime integers n and m, and we consider the curve

$$C = \{x^m = y^n\}.$$

Example. When $m = 2, n = 3$ we have $\mathcal{O}_C^{\wedge 0} = \mathbb{C}[\![t^2, t^3]\!]$, from here we can check that

$$\mathrm{Hilb}^k(C, 0) = \{(t^k + \lambda t^{k+1}), (t^{k+1}, t^{k+2})\} = \mathbb{P}^1, k \geq 2.$$

$\lambda \in \mathbb{C}$ "$\lambda = \infty$"

From now on, we fix coprime integers n and m, and we consider the curve

$t^2 = y \subset I_k/xI_k$

$\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$

$$C = \{x^m = y^n\}.$$

Example. When $m = 2, n = 3$ we have $\mathcal{O}_C^{\wedge 0} = \mathbb{C}[\![t^2, t^3]\!]$, from here we can check that

$$\mathrm{Hilb}^k(C,0) = \{(t^k + \lambda t^{k+1}), \underbrace{(t^{k+1}, t^{k+2})}\} = \mathbb{P}^1, k \geq 2.$$

$I_k = (t^k)$

$t^k, t^{k+2}, t^{k+3}, \dots$

Definition. We will write

$I_k = (t^{k+1}, t^{k+2})$

$y \subset I_k/xI_k$

$\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$

$$\mathrm{Hilb}(C,0) := \coprod_{k\geq 0} \mathrm{Hilb}^k(C,0).$$

$t^{k+3}, t^{\dots}$

xI_k $\quad$ I_k/xI_k

$t^3 I^k$ $\quad$ $\mathrm{span}\{t^k, t^{k+2}, t^{k+?}\}$

$$C = \{x^n = y^n\}$$

Note that for each ideal $I \subseteq \mathcal{O}_C^{\wedge 0}$, we have $\dim(I/xI) = n$. So we can define the *parabolic Hilbert scheme:*

$$\mathrm{PHilb}_x^{k,k+n}(C,0) \coloneqq \{I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k \mid I_d \in \mathrm{Hilb}^d(C,0)\}.$$

ideals

$$\mathrm{PHilb}_x(C,0) \coloneqq \coprod_{k \geq 0} \mathrm{PHilb}_x^{k,k+n}(C,0).$$

$$\begin{array}{ccc} \mathrm{PHilb}^k & & I_k \supseteq \cdots \supseteq xI_k \\ \downarrow & & \downarrow \\ \mathrm{Hilb}^k & & I_k \end{array}$$

$\gamma_i \geq 0$ $\quad \sum \gamma_i = n$

Moreover, if $\gamma = (\gamma_1, \dots, \gamma_r)$ is a weak composition of n, we can define

$$\mathrm{PHilb}_x^{\gamma}(C,0) := \{ J^0 \overset{\gamma_1}{\supseteq} J^1 \overset{\gamma_2}{\supseteq} \cdots \overset{\gamma_3}{\supseteq} J^r = xJ^0 \mid J^d \in \mathrm{Hilb}(C,0), \dim\left(J^{i-1}/J^i\right) = \gamma_i \}.$$

$\mathrm{PHilb}_x^{(1,1,\dots,1)}(C,0) = \mathrm{PHilb}_x$

$\mathrm{PHilb}^{(n)}(C,0) = \mathrm{Hilb}(C,0)$

And, if $\mathcal{C}_r(n)$ denotes all weak compositions of n with <u>exactly</u> r parts (some of which may be 0) we have the *compositional parabolic Hilbert scheme*

$$\mathrm{CPHilb}_x^r(C,0) := \coprod_{\gamma \in \mathcal{C}_r(n)} \mathrm{PHilb}_x^{\gamma}(C,0).$$

Remark: We have chosen the projection to the x**-**axis to define our parabolic Hilbert schemes. We can also choose the projection to the y-axis and we have

$$\mathrm{PHilb}_y(C,0) := \coprod_{k\geq 0} \mathrm{PHilb}_y^{k,k+m}(C,0)\,,$$

$$\mathrm{CPHilb}_y^r(C,0) := \coprod_{\gamma\in\mathcal{C}_r(m)} \mathrm{PHilb}_y^{\gamma}(C,0)\,.$$

Remark: We have chosen the projection to the x-axis to define our parabolic Hilbert schemes. We can also choose the projection to the y-axis and we have

$$\mathrm{PHilb}_y(C,0) := \coprod_{k\geq 0} \mathrm{PHilb}_y^{k,k+m}(C,0)\,,$$

$$\mathrm{CPHilb}_y^r(C,0) := \coprod_{\gamma\in\mathcal{C}_r(m)} \mathrm{PHilb}_y^{\gamma}(C,0)\,.$$

Remark: We have an action of the 1-dimensional torus $\mathbb{C}^*$ on the curve C:

Note: 0 is fixed

$$t.(x,y) = (t^n x, t^m y).$$

This action lifts to all the Hilbert schemes we are considering.

Theorem (Gorsky-S.-Vazirani). Recall that $C = \{x^m = y^n\}$ with m and n coprime.

a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}\,(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

b) There is an action of the *spherical rational Cherednik algebra* $\mathbf{e}H_{1,m/n}\mathbf{e}$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C,0))$.

c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n,r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C,0))$.

In all these cases, the corresponding representation is an *irreducible* and *highest weight* module that can be completely identified.

II. The algebras

(a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

Let $n \geq 0, t, c \in \mathbb{C}$. We define the *rational Cherednik algebra* $H_{t,c}(n) = H_{t,c}(S_n, \mathbb{C}^n)$ as the quotient of the semidirect product algebra

$$\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle \rtimes S_n$$

$w x_i = x_{w(i)} w$
$w y_i = y_{w(i)} w$

by the relations

- $[x_i, x_j] = 0 = [y_i, y_j]$.
- $[y_i, x_j] = c(ij), \quad i \neq j$.
- $[y_i, x_i] = t - c\sum_{j\neq i}(ij)$.

e.g. $t = 0 = c$

$\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes S_n$

$t = 1, c = 0$

$\mathcal{D}(\mathbb{C}^n) \rtimes S_n$

(a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

Another way to define the algebra $H_{1,c}(n)$ is via its *polynomial representation.* The algebra $H_{1,c}(n)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n])$ generated by:

$H_{1,c} \curvearrowright \mathbb{C}[x_1, \ldots, x_n]$

- Elements of S_n.
- Multiplication by $x_1, \ldots, x_n$. $\frac{\partial}{\partial x_i}$
- *Dunkl operators* $y_i = \partial_i - c \sum_{j \neq i} \frac{1}{x_i - x_j}(1 - (ij))$.

Fact: the polynomial representation has a unique irreducible quotient. When $c = m/n$, $\mathrm{gcd}(\mathrm{m}, \mathrm{n}) = 1$, we will denote it by $L_{m/n}$.

$L_{\frac{m}{n}}(n)$

Action

(a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

- The action of S_n is via correspondences, in a Springer-like way.

$\mathrm{PHilb}^{k,k+n}$

$\downarrow$

$\vdash S_{n-i}$ $\qquad \mathrm{PHilb}^{k,k+n,(i)}$ ← skip the i^{th} ideal

$I_k \supseteq I_{k+1} \supseteq .. \supseteq I_{k+i-1} \supseteq I_{k+i+1} \supseteq ..$

- We have a map $T\colon \mathrm{PHilb}_x^{k,k+n} \to \mathrm{PHilb}_x^{k+1,k+1+n}$, $(I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k) \mapsto (I_{k+1} \supseteq \cdots \supseteq I_{k+n} = xI_k \supseteq I_{k+1+n} = xI_{k+1})$.

T is injective and identifies $\mathrm{PHilb}^{k,k+n}$ with its image

- The image of T is the zero locus of a section of a line bundle, so we have a Gysin map in homology $\Lambda\colon H_*^{\mathbb{C}^*}\left(\mathrm{PHilb}_x^{k+1,k+1+n}\right) \to H_*^{\mathbb{C}^*}\left(\mathrm{PHilb}_x^{k\ k+n}\right)$.

Action

(a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

We have that the action of T corresponds to $x_1(1 \cdots n)$ and the action of Λ to $(1 \cdots n)^{-1} y_1$. These, together with S_n, already generate $H_{1,m/n}(n)$.

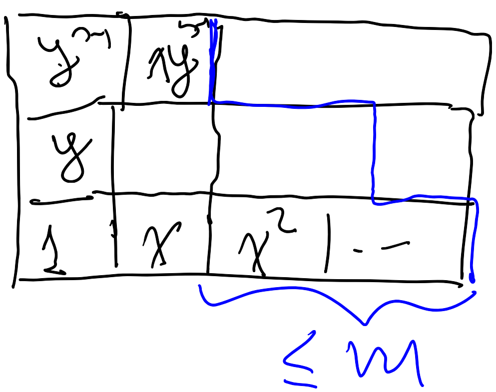
Action

(a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

We have that the action of T corresponds to $x_1(1 \cdots n)$ and the action of Λ to $(1 \cdots n)^{-1}y_1$. These, together with S_n, already generate $H_{1,m/n}(n)$.

$H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

- Fixed point basis in equivariant homology.



$L_{m/n}$.

- Basis of "non-symmetric Jack polynomials", studied by S. Griffeth.

Action

(a) There is an action of the *rational Cherednik algebra* $H_{1,m/n}(S_n, \mathbb{C}^n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

We have that the action of T corresponds to $x_1(1 \cdots n)$ and the action of Λ to $(1 \cdots n)^{-1} y_1$. These, together with S_n, already generate $H_{1,m/n}(n)$.

$H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C,0))$.

- Fixed point basis in equivariant homology. $\leftarrow (m;n) = 1$
- Intersection product with tautological line bundles on PHilb_x.

$L_{m/n}$.

- Basis of "non-symmetric Jack polynomials", studied by S. Griffeth.
- Action of Dunkl-Opdam operators
$$u_i := x_i y_i - c\underbrace{\mathrm{JM}_i}_{\sum_{j<i}(ij)}.$$

Spherical subalgebra

(b) There is an action of the *spherical rational Cherednik algebra* $\mathbf{e}H_{1,m/n}\mathbf{e}$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C,0))$.

(b) is a formal consequence of (a). Indeed, the algebra $H_{1,m/n}\,(n)$ contains the *trivial idempotent*

$$\mathbf{e} \;:=\; \frac{1}{n!}\sum_{w\in S_n} w\,,$$

and the spherical rational Cherednik algebra is, by definition, $\boxed{\mathbf{e}H_{1,m/n}\,(n)\mathbf{e}.}$
Now the result follows since $H_*^{\mathbb{C}^*}(\mathrm{Hilb}) = H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x)^{S_n} = \mathbf{e}H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x)$.

$$\begin{array}{c} \mathrm{PHilb}_x \\ \downarrow \\ \mathrm{Hilb} \end{array}$$

Quantized Gieseker algebras

(c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n,r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C,0))$.

The *Gieseker moduli space* $\mathcal{M}(n,r)$ is the moduli space of rank r torsion-free sheaves on $\mathbb{P}^2$ with fixed trivialization at infinity and second Chern class $c_2 = n$. It can be obtained via GIT Hamiltonian reduction

ADHM rels

$$\mathcal{M}(n,r) = \{(A,B,i,j) \in M_{n\times n}^2 \times M_{n\times r} \times M_{r\times n} \mid [A,B] + ij = 0\} /\!/^{\det} \mathrm{GL}_n$$

$$T^*(G^n) /\!/\!/_0^{\det} GL_n$$

The algebra $\mathcal{A}_c(n,r)$ is a quantization of the algebra of global functions on $\mathcal{M}(n,r)$. It depends on a parameter $c \in \mathbb{C}$.

$$R = \mathrm{Rep}\left(\begin{array}{c} 1 \\ \downarrow \\ n \circlearrowleft \end{array}\right)$$

Quantized Gieseker algebras

(c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n,r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C,0))$.

The algebra $\mathcal{A}_c(n,r)$ may be defined via quantum Hamiltonian reduction.

$$\mathcal{A}_c(n,r) = \left[\frac{D(R)}{D(R)\{\xi_R - c\,\mathrm{tr}(\xi) \mid \xi \in \mathfrak{gl}_n\}}\right]^{\mathrm{GL}_n}.$$

Quantized Gieseker algebras

(c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n,r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C,0))$.

The algebra $\mathcal{A}_c(n,r)$ may be defined via quantum Hamiltonian reduction.

$$\mathcal{A}_c(n,r) = \left[\frac{D(R)}{D(R)\{\xi_R - c\operatorname{tr}(\xi) \mid \xi \in \mathfrak{gl}_n\}}\right]^{\mathrm{GL}_n}.$$

Examples.

$\mathcal{M}(n,1) = \mathrm{Hilb}^n(\mathbb{C}^2)$

- $\mathcal{A}_c(n,1) = \mathbf{e}H_{1,c}(n)\mathbf{e}$. (Gan-Ginzburg, 2004).

$\mathcal{M}(1,r) = T^*(\mathbb{P}^{r-1} \times \mathbb{C})$

- $\mathcal{A}_c(1,r) = D_c(\mathbb{P}^{r-1} \times \mathbb{C})$.

Quantized Gieseker algebras

(c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n, r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C, 0))$.

Problem: For general n, r there is no known presentation of $\mathcal{A}_{m/n}(n, r)$ by generators and relations.

Quantized Gieseker algebras

(c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n,r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C,0))$.

Problem: For general n, r there is no known presentation of $\mathcal{A}_{m/n}(n,r)$ by generators and relations.

Theorem (Etingof-Krylov-Losev-S., 2020) The vector space

$$\mathcal{L}_{m/n}(n,r) := \boxed{\left(L_{n/m}(m) \otimes (\mathbb{C}^r)^{\otimes m}\right)^{S_m}}$$

explain why y instead of x

is an irreducible representation of $\mathcal{A}_{m/n}(n,r)$.

Since $L_{n/m}(m) = H_*^{\mathbb{C}^*}(\mathrm{PHilb}_y)$, this implies (c).

$$\bigoplus_{\gamma \in Cr(m)} \underbrace{L_{n/m}(m)^{S_\gamma}}_{H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^\gamma)} \qquad S_\gamma \subseteq S_n$$

Weight spaces

(c) There is an action of the *quantized Gieseker algebra* $\mathcal{A}_{m/n}(n,r)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C,0))$.

There is a *quantum comoment map* $\mathfrak{gl}_r \to \mathcal{A}_{m/n}(n,r)$ so every $\mathcal{A}_{m/n}(n,r)$-module becomes a $\mathfrak{gl}_r$-representation. The representation $\mathcal{L}_{m/n}(n,r)$ is $\mathfrak{gl}_r$-locally finite and we can geometrically describe the weight spaces.

Indeed, we can see $\gamma \in \mathcal{C}_r(m)$ as a $\mathfrak{gl}_r$-weight, and

$$\mathcal{L}_{m/n}(n,r)^\gamma = H_*^{\mathbb{C}^*}(\mathrm{PHilb}_y^\gamma(C,0)).$$

We can fix n and let $m \to \infty$ and we get the following result.

Theorem (Gorsky-S.-Vazirani) Consider the *non-reduced* curve $C_0 = \{y^n = 0\}$. Then:

$H_{0c} \cong H_{0,1}$

a) There is an action of the rational Cherednik algebra $H_{0,1}(n)$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_x(C_0, 0))$ affording the polynomial representation of $H_{0,1}(n)$.

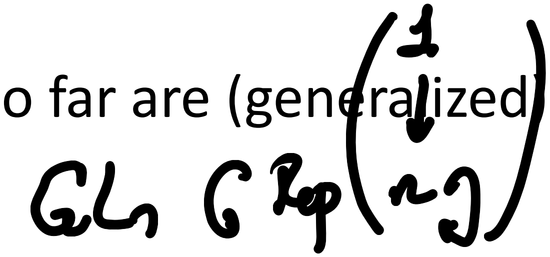
b) There is an action of the spherical rational Cherednik algebra $\mathbf{e}H_{0,1}(n)\mathbf{e}$ on the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C_0, 0))$ affording the polynomial representation of $\mathbf{e}H_{0,1}(n)\mathbf{e}$.

III. Generalized affine Springer fibers and Coulomb branches

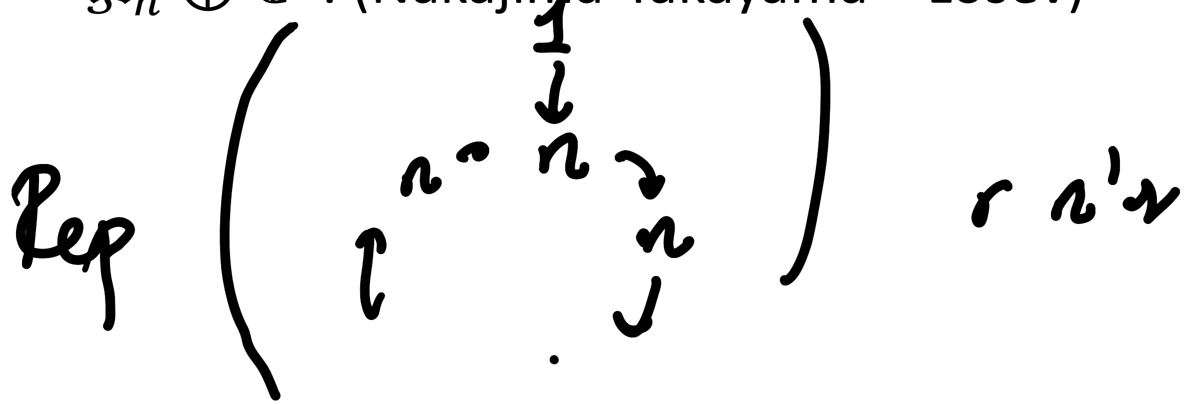
Given a reductive group G acting on a vector space N, Braverman-Finkelberg-Nakajima define a Poisson algebra $\mathcal{R}(G,N)$ whose spectrum is known as a $Coulomb\ branch$. The algebra $\mathcal{R}(G,N)$ admits a natural quantization $\mathcal{R}_\hbar(G,N)$ that we call a Coulomb branch algebra.

BFN's construction was generalized by Webster to account for non-trivial line defects. Besides G and N, this algebra depends on a parahoric subgroup $P \subseteq G[[t]]$ and a nice subspace $L \subseteq N((t))$ that is stabilized by L. BFN's construction is the special case $P = G[[t]]$ and $L = N[[t]]$.

All of the algebras that have appeared so far are (generalized) Coulomb branch algebras.



- The spherical rational Cherednik algebra $\mathbf{e}H_{1,c}(n)\mathbf{e}$ is the Coulomb branch for $G = \mathrm{GL}_n$ acting on $N = \mathfrak{gl}_n \oplus \mathbb{C}^n$. (Kodera-Nakajima, Webster)
- The full Cherednik algebra $H_{1,c}(n)$ appears with the same G and N as above, but choosing $P = I$, the standard Iwahori, and $L = \mathfrak{i} \oplus \mathbb{C}[[t]]^n$. (Webster)
- The quantized Gieseker algebra $\mathcal{A}_c(n,r)$ is the Coulomb branch for $G = \mathrm{GL}_n^r$ acting on $N = \mathfrak{gl}_n^r \oplus \mathbb{C}^n$. (Nakajima-Takayama + Losev)



BFN Springer theory

Let G, N and $P \subseteq G[[t]], L \subseteq N((t))$ be as above. For $v \in N((t))$ we define the *generalized affine Springer fiber*

$$\mathrm{Spr}_{L,P}(v) := \{[g] \in G((t))/P \mid g^{-1}v \in L\}.$$

So that, when $P = G((t))$, $\mathrm{Spr}_{L,P}(v) \subseteq \mathcal{G}r_G$ and when $P = I$, $\mathrm{Spr}_{L,P}(v) \subseteq \mathcal{F}\ell_G$.

Theorem (Hilburn-Kamnitzer-Weekes, Garner-Kivinen 2020) Under mild assumptions on v, the (generalized) Coulomb branch algebra acts on $H_*^{\mathbb{C}^*}(\mathrm{Spr}_{L,P}(v))$.

All of our Hilbert schemes are generalized affine Springer fibers. Let

action of y on $\mathcal{O}_C^{\wedge 0}$

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & t^m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

All of our Hilbert schemes are generalized affine Springer fibers. Let

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & t^m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$\begin{tikzcd} \mathbb{C}[[t]] \arrow[d, "b"] \\ \mathbb{C}^n[[t]] \arrow[loop left, "Y"] \end{tikzcd}$$

Then,

- $\mathrm{Hilb}(C,0) = \mathrm{Spr}_{N[[t]],G[[t]]}(Y,b)$ (Garner-Kivinen).
- $\mathrm{PHilb}_x(C,0) = \ \mathrm{Spr}_{\mathfrak{i}\oplus\mathbb{C}[[t]]^n,\, I}(Y,b)$ (Garner-Kivinen, Gorsky-S.-Vazirani).
- $\mathrm{CPHilb}^r_y(C,0) = \mathrm{Spr}_{N[[t]],G[[t]]}((\mathbf{1},\mathbf{1},\ldots,\mathbf{1},Y),b)$ (Gorsky-S.-Vazirani).

$$\begin{tikzcd} & \mathbb{1} \arrow[d, "b"] \\ \arrow[d, "Y"] & \mathbb{C}^n \\ \mathbb{C} & \end{tikzcd}$$

Note: We did not need the fact that m and n are coprime to identify Hilbert schemes with generalized affine Springer fibers. So BFN Springer theory allows us to conclude that Cherednik/Gieseker algebras act on equivariant homology of Hilbert schemes even in the non-coprime case.

Note: We did not need the fact that m and n are coprime to identify Hilbert schemes with generalized affine Springer fibers. So BFN Springer theory allows us to conclude that Cherednik/Gieseker algebras act on equivariant homology of Hilbert schemes even in the non-coprime case.

Problem: Describe the corresponding representation. It is no longer irreducible.

Note: We did not need the fact that m and n are coprime to identify Hilbert schemes with generalized affine Springer fibers. So BFN Springer theory allows us to conclude that Cherednik/Gieseker algebras act on equivariant homology of Hilbert schemes even in the non-coprime case.

Problem: Describe the corresponding representation. It is no longer irreducible.

Conjecture: This representation is completely reducible and has minimal support (i.e., its annihilator is the unique maximal two-sided ideal).

Evidence for the conjecture:

- Character formulas of Etingof-Gorsky-Losev match conjectures of Oblomkov-Rasmussen-Shende.
- The EKLS theorem relating representations of quantized Gieseker varieties with those of RCA is valid for representations with minimal support.

Moreover, ((compositional) parabolic) punctual Hilbert schemes on plane curves are generalized affine Springer fibers.

Moreover, ((compositional) parabolic) punctual Hilbert schemes on plane curves are generalized affine Springer fibers.

$C_0 = \{y^n = 0\}$

Recall that $C_0 = \{y^n = 0\}$. According to BFN Springer theory, the equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}_y^r(C_0, 0))$ should have an action of the algebra of functions on

$$\mathcal{M}'(n,r) = \{(A,B,i,j) \in M_{n\times n}^2 \times M_{n\times r} \times M_{r\times n} \mid [A,B] + ij = \mathbf{1}\} \big/ \mathrm{GL}_n$$

Is this in any way related to the representation of the Cherednik algebra on $H_*^{\mathbb{C}^*}(\mathrm{PHilb})$?

Thanks for your attention!