Wall-Crossing for Newton-Okounkov bodies

joint work with Megumi Harada

arXiv: 1912.04809.

Outline

I. Polytopes & algebraic geometry

II. Newton-Okounkov bodies (NO bdy)

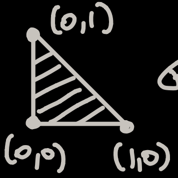
III. Wall-crossing for Newton-Okounkov bodies

IV. The tropical Grassmannian

The Newton polytope of $f = \sum_a c_a x^a \in \mathbb{C}[x_1, \dots, x_n]$ is

$$\mathrm{Newt}(f) := \mathrm{conv}\{ a \mid c_a \neq 0 \}.$$

Example: $\begin{cases} ax + by + c = 0 \\ dx + ey + f = 0 \end{cases}$

$\mathrm{Newt}(3x + y - 1) =$ 

$$1 = 2!\,\mathrm{vol}(\triangle).$$

Bernstein-Khovanskii-Kuchnirenko Thm:

$$A = \{a_1, \dots, a_s\} \subseteq \mathbb{Z}^n$$

$$L_A = \left\{ \sum_{i=1}^{s} c_i x^{a_i} \,\middle|\, c_i \in \mathbb{C} \right\}$$

For a generic choice of $f_1, \dots, f_n \in L_A$, the number of solutions to $f_1(x) = \dots = f_n(x) = 0$ in $(\mathbb{C}^*)^n$ is equal to $n!\,\mathrm{vol}(\mathrm{conv}(A))$.

II. NO-bodies

Δ a polytope $\rightsquigarrow X_\Delta \subseteq \mathbb{P}^s$ a projective toric variety

Thm: $\deg(X_\Delta) = n!\,\mathrm{vol}(\Delta)$

NO-bodies: input: $X \subseteq \mathbb{P}^s$ a projective variety
+ a valuation on its coord ring

output: a convex body Δ st $\deg(X) = n!\,\mathrm{vol}(\Delta)$

[Okounkov, Lazarsfeld-Mustaţă, Kaveh-Khovanskii]

Valuations

Equip $\mathbb{Z}^n$ with a total order $\succsim$

$\nu : \mathbb{C}[X] \setminus \{0\} \longrightarrow \mathbb{Z}^n$ is a valuation if

i) $\nu(f+g) \succsim \max\{\nu(f), \nu(g)\}$

ii) $\nu(fg) = \nu(f) + \nu(g)$

iii) $\nu(c) = 0$ for all $c \in \mathbb{C} \setminus \{0\}$

Valuations

Equip $\mathbb{Z}^n$ with a total order $\succsim$

$\nu : \mathbb{C}[X] \setminus \{0\} \longrightarrow \mathbb{Z}^n$ is a valuation if
i) $\nu(f+g) \succsim \max\{\nu(f), \nu(g)\}$
ii) $\nu(fg) = \nu(f) + \nu(g)$
iii) $\nu(c) = 0$ for all $c \in \mathbb{C} \setminus \{0\}$

Example: $\nu\left(\sum_a c_a X^a\right) := \min\{a \mid c_a \neq 0\}$ is a valuation on $\mathbb{C}[x_1, \dots, x_n]$

NO-body $\Delta(X, \nu) := \overline{\mathrm{cone}(\mathrm{image}(\nu))} \cap \{x_1 = 1\}$

Thm [Okounkov, Lazarsfeld-Mustaţă, Kaveh-Khovanskii]: let X be n-dim'l as ν as above $\Rightarrow \deg(X) = n! \, \mathrm{vol}(\Delta(X, \nu))$.

An example

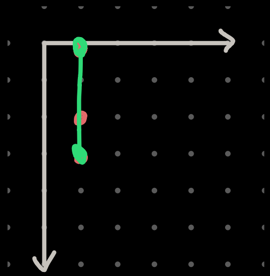
X the hypersurface given by $I = \langle y^2z - x^3 + 7xz^2 - 2z^3 \rangle$.

let ν be the following valuation:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix} \rightsquigarrow \nu(x^\alpha y^\beta z^\gamma) = M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\nu\left(\sum c_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma\right) = \min\{\nu(x^\alpha y^\beta z^\gamma) \mid c_{\alpha\beta\gamma} \neq 0\}$$

image(ν) = semigroup generated by the columns of M



$\Delta(X, \nu)$ = convex hull of the columns of M

Anderson: When $\Delta(X,\nu)$ is a rational polytope, there is a degeneration of X to the toric variety of $\Delta(X,\nu)$.

i.e. a one-parameter flat family $\pi: \mathfrak{X} \longrightarrow \mathbb{A}^n$ with $\pi^{-1}(1) = X$ and $\pi^{-1}(0) = X_{\Delta(X,\nu)}$.

Kaveh-Manon: construct valuations with $\Delta(X,\nu)$ a rational polytope using tropical geometry.

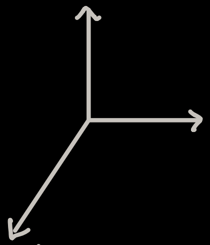
The tropicalization of I is $\operatorname{trop}(I) := \{w \in \mathbb{Q}^n \mid \operatorname{in}_w I \text{ contains no monomial}\}$

$\operatorname{trop}(I)$ has the structure of a fan:

the open cones are $C_w = \{w' \in \operatorname{trop}(I) \mid \operatorname{in}_{w'} I = \operatorname{in}_w I\}$.

Example: $I = \langle y^2 z - x^3 + 7xz^2 - 2z^3 \rangle$

$\text{cone}(\pm(1,1,1),(0,1,0)) \leftrightarrow \text{in}_w I = \langle -x^3 + 7xz^2 - 2z^3 \rangle$

$\text{trop}(I) =$  $\text{cone}(\pm(1,1,1),(1,0,0)) \leftrightarrow \text{in}_w I = \langle y^2 z - 2z^3 \rangle$

$\text{cone}(\pm(1,1,1),(-2,-3,0)) \leftrightarrow \text{in}_w I = \langle y^2 z - x^3 \rangle$

A cone C in $\text{trop}(I)$ is prime if $\text{in}_w I$ is prime for some/any $w \in C^\circ$.

Let C be a prime cone of $\text{trop}(I)$ and $\{u_1, \dots, u_n\} \subseteq C$ be maximally linearly independent. Kaveh-Manon construct a valuation $\nu : \mathbb{C}[x_1, \dots, x_s]/I \longrightarrow \mathbb{Z}^n$ whose NO-body is a rational polytope

Let C be a prime cone of $\operatorname{trop}(I)$ and $\{u_1, \dots, u_n\} \subseteq C$ be maximally linearly independent. Kaveh-Manon construct a valuation $\nu : \mathbb{C}[x_1, \dots, x_s]/I \longrightarrow \mathbb{Z}^n$ whose NO-body is a rational polytope

Explicitly, $\Delta(V(I), \nu)$ is the convex hull of the columns of the matrix $\begin{bmatrix} \text{---}\; u_1 \;\text{---} \\ \vdots \\ \text{---}\; u_n \;\text{---} \end{bmatrix}$

Example: $I = \langle y^2 z - x^3 + 7xz^2 - 2z^3 \rangle$

$$C = \operatorname{cone}\left(\pm(1,1,1), (-2,-3,0)\right) \Rightarrow M = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

$$\Delta(X, \nu) = \operatorname{conv}\{(1,-2), (1,-3), (1,0)\}$$

Geometric Wall-Crossing for NO-bodies

Thm [E.-Harada]: Let C_1, C_2 be two prime cones of $\mathrm{trop}(I)$ s.t.

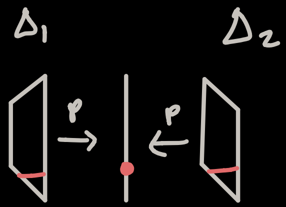
① C_1, C_2 are of maximal dimension

② C_1, C_2 share a codim 1 face

$$M_1 = \begin{bmatrix} \text{equal} \\ \hline \neq \end{bmatrix} \quad \begin{bmatrix} \text{equal} \\ \hline \neq \end{bmatrix} = M_2$$

Let $\{u_{i1}, \dots, u_{id}\} \subseteq C_i$ be maximally linearly ind. and such that $u_{11} = u_{21}, \dots, u_{1d-1} = u_{2d-1}$, $u_{1d} \neq u_{2d}$.

Then ① the fibers of the projection $p: \mathbb{R}^d \to \mathbb{R}^{d-1}$ restricted to $\Delta(X, \nu_1)$ & $\Delta(X, \nu_2)$ are line segments of the same length.



Geometric Wall-Crossing for NO-bodies

Thm [E.-Harada]: Let C_1, C_2 be two prime cones of $\mathrm{trop}(I)$ s.t.

① C_1, C_2 are of maximal dimension

② C_1, C_2 share a codim 1 face

Let $\{u_{i1}, \dots, u_{i}\} \subseteq C_i$ be maximally linearly ind. and such that

$u_{11} = u_{21}, \dots, u_{1d-1} = u_{2d-1}, \ u_{1d} \neq u_{2d}$.

Then ② We obtain 2 maps $\Delta(X, \nu_1) \longrightarrow \Delta(X, \nu_2)$

one that shifts the interval

& another that vertically reflects $\Delta(X, \nu_1)$ & then shifts.

Remarks:

① Ilten-Manon observed in 2017 that this geometric wall-crossing can be derived from the theory of complexity-one T-varieties

② Ilten interprets geometric wall-crossing as a generalization of the "combinatorial mutation" of Akhtar-Coates-Galkin-Kaprzyk.

③ E.-Hanada give an "algebraic wall-crossing", i.e. a map between the semigroups image(ν_1), image(ν_2).

Idea of proof:

① Since C_i is prime & max'l dimensional $\Rightarrow$ $\mathrm{in}_{C_i} I$ is toric,

& the variety for this ideal is the toric variety of $\Delta(X, \nu_i)$.

Let X_i = TV of $\Delta(X, \nu_i)$ & Y = variety of $\mathrm{in}_{C_1}\mathrm{in}_{C_2} I$.

③ There is a codim-1 torus $T \curvearrowright Y$

This torus is a subtorus of the torus $T_i \curvearrowright X_i$

The TV of a fiber $p^{-1}(\xi) \in \Delta(X, \nu_i)$ is the GIT quotient

of X_i by T at ξ.

② There exist flat families $\mathcal{X}_1, \mathcal{X}_2$ over $\mathbb{A}^1$ w/ generic fibers isom to Y & special fibers isomorphic to X_1, X_2 (respectively).

(4) T-action extends to the families $\mathcal{X}_1, \mathcal{X}_2$

By taking GIT quotient of $\mathcal{X}_i$ by T at ξ we obtain

2 flat families w/ generic fiber $Y /\!\!/_{\xi} T$

special fibers $X_i /\!\!/_{\xi} T$ (resp.)

(5) flat families preserve degree

$$\Rightarrow \quad \underset{\substack{\parallel \\ \text{length of fiber} \\ \text{of } \xi \text{ in} \\ \Delta(X, \nu_1)}}{\deg\left(X_1 /\!\!/_{\xi} T\right)} = \deg\left(Y /\!\!/_{\xi} T\right) = \underset{\substack{\parallel \\ \text{length of fiber} \\ \text{of } \xi \text{ in} \\ \Delta(X, \nu_2)}}{\deg\left(X_2 /\!\!/_{\xi} T\right)}$$

Case Study: The tropical Grassmannian $Gr(2,m)$

$Gr(2,m)$ = space of 2-planes in $\mathbb{C}^m$

$I_{2,m}$ = plücker ideal of $Gr(2,m)$

$I_{2,4} = \langle P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} \rangle$

☆ All maximal cones of $\text{trop}(I_{2,4})$ are prime.

Remark: $\mathbb{C}[Gr(2,m)]$ is a cluster algebra

Roughly: $Gr(2,m)$ has an atlas of tori

transition maps between the tori are cluster mutations

Our flip wall-crossing is a tropicalized toric mutation.

Thank you!

¡Muchas gracias!