Symplectic embeddings via algebraic capacities

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Main references:

- *Algebraic capacities*, B. Wormleighton (2020)
- *ECH embedding obstructions for rational surfaces*, J. Chaidez & B. Wormleighton (2020)

Symplectic embeddings

- A central question in symplectic geometry is, given two symplectic manifolds (X, ω) and (X', ω') of the same dimension $2n$, when does there exist a *symplectic embedding* between them?
- Namely, when does there exist a smooth embedding $\iota\colon X \to X'$ such that
$$\iota^*\omega' = \omega$$
- We write $(X, \omega) \overset{\mathsf{s}}{\hookrightarrow} (X', \omega')$ when such an embedding exists.

- The first constraint we can find is *volume*:

$$(X,\omega) \overset{\mathsf{s}}{\hookrightarrow} (X',\omega') \Longrightarrow \mathrm{vol}(X,\omega) \leq \mathrm{vol}(X',\omega')$$

where $\mathrm{vol}(X,\omega) := \int_X \omega^n$.

- However the volume constraint is not very close to being *sharp* in general.

Theorem (Gromov nonsqueezing)

Let $B^{2n}(r)$ be a $2n$-dimensional ball of radius r and let $Z^{2n}(R)$ be a $2n$-dimensional cylinder of cross-sectional radius R. Then

$$B^{2n}(r) \xhookrightarrow{s} Z^{2n}(R) \text{ if and only if } r \leq R$$

- As a result, to study symplectic embeddings more closely we will require more nuanced obstructions than the volume.

Example

Define the *Gromov width*

$$c_G(X,\omega) := \sup\{r > 0 : B^{2n}(r) \overset{\mathsf{s}}{\hookrightarrow} (X,\omega)\}$$

- To give a sense of how symplectic embedding problems meet algebraic geometry, we consider the following construction of McDuff–Polterovich:
- Suppose $\amalg_{i=1}^{N} B^4(r_i) \overset{s}{\hookrightarrow} \mathbb{P}^2$. Excise the interiors of these balls and collapse their boundaries via the Hopf map to get N copies of $\mathbb{P}^1$. This is just an N point blowup.
- The existence of such an embedding is equivalent to the divisor $H - \sum r_i E_i$ on the N point blowup of $\mathbb{P}^2$ being representable by a symplectic form (e.g. ample).

- There are many other connections between symplectic embeddings and algebraic geometry, for instance:
  - Various authors relate the Gromov width of polarised varieties to Seshadri constants
  - Brendel–Mikhalkin–Schlenk construct ‘exotic’ embeddings of cubes via toric degenerations of del Pezzo surfaces
  - ...

Outline

For the remainder of this talk we will:

- discuss more general obstructions to symplectic embeddings ('symplectic capacities')
- discuss analogs in algebraic geometry ('algebraic capacities')
- use this algebraic perspective to solve problems.

Symplectic capacities

Let $\mathcal{S}$ be some class of symplectic spaces of dimension $2n$ including the ball $B^{2n}(r)$. We say that a function $c\colon \mathcal{S} \to \mathbb{R}$ is a *capacity* if:

- $(X, \omega) \overset{\mathsf{s}}{\hookrightarrow} (X', \omega') \implies c(X, \omega) \leq c(X', \omega')$
- $c(X, \alpha\omega) = \alpha \cdot c(X, \omega)$
- $c(B^{2n}(1)) > 0$.

Some contact geometry

Consider a $(2n-1)$-manifold Z.

- A *contact form* on Z is a 1-form λ with $\lambda \wedge (d\lambda)^{n-1} > 0$
- The *Reeb vector field* is determined by $\begin{cases} d\lambda(R, -) = 0 \\ \lambda(R) = 1 \end{cases}$
- A *Reeb orbit* is a closed orbit of R; that is, a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Z$ with $\gamma'(t) = R(\gamma(t))$

Example

Define the ellipsoid with symplectic radii a, b to be

$$E(a,b) := \{(z_1, z_2) \in \mathbb{C}^2 : \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\}$$

We often write $z_j = x_j + iy_j$.

Example

Let $Z = \partial E(a, b)$.

- A contact form on Y is the restriction of the Liouville form

$$\lambda = \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j)$$

- The corresponding Reeb vector field is

$$R = \frac{2\pi}{a}\frac{\partial}{\partial\theta_1} + \frac{2\pi}{b}\frac{\partial}{\partial\theta_2}$$

- If $a/b \notin \mathbb{Q}$ then the only embedded Reeb orbits are

$$\gamma_1 = (z_2 = 0) \text{ and } \gamma_2 = (z_1 = 0)$$

- We consider an infinite-dimensional graded $\mathbb{Z}/2$-vector space $\mathrm{ECH}_*(\partial E(a,b))$ associated to $\partial E(a,b)$:

$$\begin{tikzcd}
\mathbb{Z}/2 & 0 \arrow[l, phantom] & \mathbb{Z}/2 \arrow[ll, bend left] & 0 \arrow[ll, bend right] & \mathbb{Z}/2 \arrow[ll, bend left] & 0 \arrow[ll, bend right] & \mathbb{Z}/2 \arrow[ll, bend left] & \cdots \arrow[ll, bend right]
\end{tikzcd}$$

- ...with a (surjective) degree -2 map U defined via curve counting
- ...and with a filtration $\mathrm{ECH}_*(\partial E(a,b))^{\leq L}$ from the 'action'

$$\mathcal{A}(\gamma) = \int_\gamma \lambda$$

- $\mathrm{ECH}_*(\partial E(a,b))$ has a generator Γ_k in each even degree $2k$ represented by
$$\{(\gamma_1, m_1), (\gamma_2, m_2)\}$$
for a pair of nonnegative integers (m_1, m_2).
- The action is $\mathcal{A}\{(\gamma_1, m_1), (\gamma_2, m_2)\} = m_1 a + m_2 b$.
- Since the action increases with degree, we have
$$\begin{aligned}\mathcal{A}(\Gamma_k) &= N_k(a,b) \\ &:= k\text{th smallest number of the form } m_1 a + m_2 b\end{aligned}$$
where $\Gamma_k = \{(\gamma_1, m_1), (\gamma_2, m_2)\}$ is the generator of degree $2k$.

- For each contact 3-manifold (Z, λ) there is a graded vector space $\mathrm{ECH}_*(Z, \lambda)$ with the structure above called the *Embedded Contact Homology* of of (Z, λ).
- Define

$$c_k^{\mathsf{ech}}(Z, \lambda) := \inf\{L > 0 : \exists \Gamma \in \mathsf{ECH}_*(Z, \lambda)^{\leq L} \text{ with } U^k\Gamma = [\emptyset]\}$$

- Thus

$$c_k^{\mathsf{ech}}(\partial E(a, b)) = N_k(a, b)$$

- For a symplectic manifold (X, ω) define

$$c_k^{\mathsf{ech}}(X, \omega) := c_k^{\mathsf{ech}}(\partial X, \lambda)$$

where $d\lambda = \omega|_{\partial X}$.

- Thus

$$c_k^{\mathsf{ech}}(E(a, b)) = N_k(a, b)$$

Two big results

- $E(a,b) \overset{\mathsf{s}}{\hookrightarrow} E(c,d)$ iff $c_k^{\mathsf{ech}}(E(a,b)) \leq c_k^{\mathsf{ech}}(E(c,d))$ for all k. (Hofer conjecture / McDuff's theorem 2010)
- ECH capacities asymptotically return the volume constraint:

$$\lim_{k\to\infty} \frac{c_k^{\mathsf{ech}}(X,\omega)^2}{k} = 4\,\mathrm{vol}(X,\omega)$$

(Cristofaro-Gardiner–Hutchings–Ramos 2015)

Example

- Let $\Delta_{a,b}$ be the triangle with vertices $(0,0), (\frac{1}{a}, 0), (0, \frac{1}{b})$.
- Observe that lattice points in $r\Delta_{a,b}$ are pairs of integers (m_1, m_2) such that

$$am_1 + bm_2 \leq r$$

- This is exactly

$$\operatorname{rk} \mathrm{ECH}_*(\partial E(a,b), \lambda)^{\leq r}$$

- When $a, b \in \mathbb{Q}$ this is given by a quasi-polynomial for $r \in \mathbb{Z}_{\geq 0}$ by Ehrhart's theorem.

Example

- Let $Y = \mathbb{P}(1, a, b)$ be a weighted projective space.
- Observe that sections of $\mathcal{O}_Y(r)$ correspond to pairs of integers (m_1, m_2) such that

$$am_1 + bm_2 \leq r$$

- This is exactly

$$\operatorname{rk} \mathrm{ECH}_*(\partial E(a, b), \lambda)^{\leq r}$$

- When $a, b \in \mathbb{Q}$ this is given by a quasi-polynomial for $r \in \mathbb{Z}_{\geq 0}$ by orbifold Riemann–Roch.

- Moreover, the complement of $\mathcal{O}_Y(1)$ in $\mathbb{P}(1,a,b)$ is symplectomorphic to $E(a,b)^\circ$.
- ...

Algebraic capacities

- Consider a symplectic 4-manifold (X, ω) such that its interior X° has a compactification to a projective surface Y with boundary divisor A.
- We introduce algebraic invariants on (Y, A) to recover the embedding obstructions from ECH for (X, ω).

Definition

For a pair (Y, A) consisting of a normal projective surface Y and a $\mathbb{Q}$-Cartier big $\mathbb{R}$-divisor A define the *kth algebraic capacity*

$$c_k^{\mathsf{alg}}(Y, A) := \inf_{D \in \mathrm{Nef}(Y)} \{D \cdot A : \chi(D) \geq k + \chi(\mathcal{O}_Y)\}$$

In a situation where Noether's formula holds we have

$$c_k^{\mathsf{alg}}(Y, A) = \inf_{D \in \mathrm{Nef}(Y)} \{D \cdot A : D \cdot (D - K_Y) \geq 2k\}$$

- For $Y = \mathbb{P}^2, A = \mathcal{O}(a)$,

$$\begin{aligned} c_k^{\mathsf{alg}}(Y, A) &= \inf_{d \geq 0} \{ad : d(d+3) \geq 2k\} \\ &= 0, a, a, 2a, 2a, 2a, \ldots \end{aligned}$$

- This equals $c_k^{\mathsf{ech}}(B(a)) = N_k(a, a)$ where $B(a) = E(a, a)$

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$$= 0, a, a, 2a, 2a, 2a, \ldots$$

- This equals $c_k^{\mathsf{ech}}(B(a)) = N_k(a, a)$ where $B(a) = E(a, a)$ with $B(a, a)^\circ \simeq \nu_a(\mathbb{P}^2) \setminus H$.

Theorem (Chaidez–W. '20)

Let (Y, A) *be a polarised rational surface that is either smooth or toric. Suppose* (X, ω) *is a star-shaped domain in* $\mathbb{R}^4$*. If* $(X, \omega) \overset{\mathsf{s}}{\hookrightarrow} Y$ *then*

$$c_k^{\mathsf{ech}}(X, \omega) \leq c_k^{\mathsf{alg}}(Y, A) \text{ for all } k.$$

- In particular, for many divisor complements $Y \setminus A$ we obtain $c_k^{\mathsf{ech}}(Y \setminus A) \leq c_k^{\mathsf{alg}}(Y, A)$.

Idea of the proof

There are two key ideas in the proof:

- Neck stretching to produce Reeb orbits from divisors
- Relating Seiberg–Witten invariants to nefness

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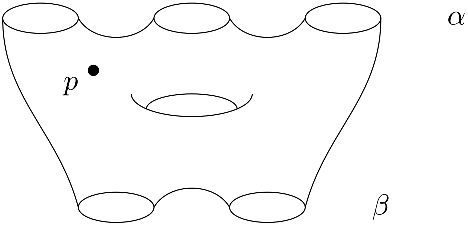
- Neck stretching to produce Reeb orbits from divisors
- Relating Seiberg–Witten invariants to nefness

Write

$$Y = X \cup_Z N$$

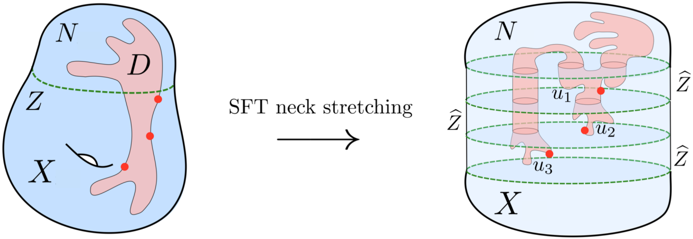
where Z is the image of ∂X.

- I need to say a little more about the U map on $\mathrm{ECH}_*(Z, \lambda)$.
- Let $\deg \alpha = k$ and let $\deg(\beta) = k - 2$. The coefficient of β in $U(\alpha)$ is the number of holomorphic curves 'of index 2' in $\widehat{Z} = Z \times \mathbb{R}$ that are positively asymptotic to α and negatively asymptotic to β, and satisfy a general point constraint:



- Thus $c_1^{\mathrm{ech}}(X)$ is computed as the minimum area of such holomorphic curves $u = \cup_i C_i$ in $\widehat{Z}$ passing through a general point p.
- Equivalently, these objects are just formal $\mathbb{Z}_{\geq 0}$-linear combinations of holomorphic curves with a global incidence constraint.
- $c_k^{\mathrm{ech}}(X)$ is given by sequences $(u_1, \ldots, u_k)$ of k such objects with matching ends; note that here there are k general point constraints.

- Suppose $D = \sum a_i D_i$ is an effective divisor in Y.
- One can produce a sequence $(u_1, \ldots, u_k)$ by *neck stretching* Y along Z so long as D passes through k general points.
- In other words, we expect $\chi(D) \geq k+1$. The area of $(u_1, \ldots, u_k)$ is $D \cdot A$.


N
D
Z
X
SFT neck stretching
N
$\widehat{Z}$
u_1
$\widehat{Z}$
u_2
$\widehat{Z}$
u_3
X

- However, not all effective divisors contribute to the infimum defining $c_k^{\text{ech}}(X)$. The divisors that do contribute have nonzero *Seiberg–Witten invariant*.
- We show that

$$\inf_{\mathrm{SW}(D)\neq 0}\{D\cdot A : \chi(D)\geq k+1\} = \inf_{D\in \mathrm{Nef}(Y)}\{D\cdot A : \chi(D)\geq k+1\}$$

using the recursive (field theory) structure of Seiberg–Witten invariants and a clever transform on $\mathrm{Pic}(Y)$.

Advantages of algebraic capacities

Algebraic capacities have several advantages

- They are often much more computable than ECH when the nef cone is reasonable
- In cases where the nef cone is poorly understood we still obtain many ('smart') obstructions of the form $c_k^{\mathsf{ech}}(X, \omega) \leq D \cdot A$
- They are suited to study 'non-generic' symplectic manifolds where symplectic techniques often break down.

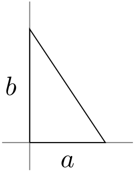
Toric manifolds

- We specialise to toric manifolds for some applications.
- Let $\mu\colon \mathbb{C}^2 \to \mathbb{R}^2$ be the moment map for $(S^1)^2 \curvearrowright \mathbb{C}^2$.
- For a connected region $\Omega \subseteq \mathbb{R}^2$ define

$$X_\Omega := \mu^{-1}(\Omega)$$

Example

If Ω is



then

$$X_\Omega = E(a, b)$$

- If $\Omega \subseteq \mathbb{R}^2_{\geq 0}$ has adjacent edges on the coordinate axes and is convex we say that X_Ω is a *convex toric domain*.
- If Ω is a lattice / rational / rational-sloped polytope we say that X_Ω is a *lattice / rational / rational-sloped convex toric domain*.

Theorem (W. '19)

Suppose X_Ω is a rational-sloped convex toric domain. Let (Y_Ω, A_Ω) be the polarised toric surface corresponding to Ω. We have

$$\begin{aligned} c_k^{\mathsf{ech}}(X_\Omega) &= c_k^{\mathsf{alg}}(Y_\Omega, A_\Omega) \\ &= \inf_{D \in \mathrm{Nef}(Y_\Omega)} \{D \cdot A_\Omega : h^0(D) \geq k+1\} \end{aligned}$$

Hence for $x \in \mathbb{Z}_{\geq 0}$,

$$\#\{k : c_k^{\mathsf{ech}}(X_\Omega) \leq x\} = \sup_{D \in \mathrm{Nef}(Y_\Omega)} \{h^0(D) : D \cdot A_\Omega \leq x\}$$

Applications

- If $\Delta \subseteq \mathbb{R}^2$ has two edges on the coordinate axes and is 'concave' we say that X_Δ is a concave toric domain.
- E.g. balls, ellipsoids, Lagrangian bidisk,...

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- If $\Delta \subseteq \mathbb{R}^2$ has two edges on the coordinate axes and is 'concave' we say that X_Δ is a concave toric domain.
- E.g. balls, ellipsoids, Lagrangian bidisk,...

Theorem (Chaidez–W. '20)

Suppose (Y, A) is a (possibly singular) polarised toric surface and that X_Δ is a concave toric domain. Then

$$X_\Delta^\circ \overset{\mathsf{s}}{\hookrightarrow} Y \iff c_k^{\mathsf{ech}}(X_\Delta) \leq c_k^{\mathsf{alg}}(Y, A)$$

- The *Gromov width* of a symplectic manifold (X, ω) is

$$c_G(X, \omega) := \sup\{r > 0 : B(r) \overset{s}{\hookrightarrow} (X, \omega)\}$$

Theorem (Chaidez–W. '20)

Suppose Ω_1, Ω_2 are two rational-sloped polygons. If $\Omega_1 \subseteq \Omega_2$ then

$$c_G(Y_{\Omega_1}) \leq c_G(Y_{\Omega_2})$$

Sketch of proof

- Note that $c_k^{\mathsf{alg}}(Y_\Omega, A_\Omega)$ is invariant under $\mathbb{Z}$-affine maps applied to Ω; hence we can assume that Ω_1 and Ω_2 do not meet the coordinate axes.
- Thus $X^\circ_{\Omega_1} \subseteq X^\circ_{\Omega_2}$ and so

$$c_k^{\mathsf{alg}}(Y_{\Omega_1}, A_{\Omega_1}) = c_k^{\mathsf{ech}}(X_{\Omega_1}) \leq c_k^{\mathsf{ech}}(X_{\Omega_2}) = c_k^{\mathsf{alg}}(Y_{\Omega_2}, A_{\Omega_2})$$

- The result follows since c_k^{alg} sharply obstruct embeddings of balls.

- This resolves a conjecture of Averkov–Hofscheier–Nill.
- Define the *lattice width* of a polygon $\Omega \subseteq \mathbb{R}^2$ by

$$\mathrm{lw}(\Omega) = \inf_{\ell \in \mathbb{Z}^n} \{ \sup_{p,q \in \Omega} \langle \ell, p - q \rangle \}$$

Corollary (Chaidez–W. '20)

Suppose Ω is a rational-sloped polygon. Then

$$c_G(Y_\Omega) \leq \mathrm{lw}(\Omega)$$

- One could also repeat this analysis with a different concave toric domain X_Ξ in place of a ball. Define the Ξ-*width*

$$c_\Xi(X,\omega) := \sup\{r > 0 : X_{r\Xi} \overset{s}{\hookrightarrow} (X,\omega)\}$$

Corollary (Chaidez–W. '20)

Suppose Ω_1, Ω_2 are two rational-sloped polygons. If $\Omega_1 \subseteq \Omega_2$ then

$$c_\Xi(Y_{\Omega_1}) \leq c_\Xi(Y_{\Omega_2})$$

To conclude, here are some emerging connections between algebraic capacities and other quantities of algebraic positivity:

- for the *Seshadri constant* $\varepsilon(A)$ of big and nef A we have

$$\varepsilon(A) \leq c_G(Y) \leq c_1^{\mathsf{alg}}(Y, A)$$

- I predict connections between $c_k^{\mathsf{alg}}(Y, A)$ and the combinatorics of *Newton–Okounkov bodies* associated to (Y, A)
- I am using various *toric ind-schemes* to treat 'irrational' convex toric domains as appropriate limits of rational-sloped convex toric domains.

Thank you!