Polytopes, wall crossings, and cluster varieties

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• Compactifications of cluster varieties and convexity (joint with Magee, Nájera Chávez) arXiv: 1912.13052
• On cluster duality, mirror symmetry and toric degenerations of Grassmannians (joint with Bossinger, Magee and Nájera Chávez), soon!
• Towards Batyrev duality for finite-type cluster varieties (joint with Magee), in preparation
• Algebraic and symplectic viewpoint on compactifications of two-dimensional cluster varieties of finite type (joint with Vianna) arXiv:2008.03265
• Some examples of Family Floer mirror (joint with Lin) soon!
Fix a lattice $N \cong \mathbb{Z}^n$, $M = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$.  
$N_\mathbb{R} = N \otimes \mathbb{R}$, $M_\mathbb{R} = M \otimes \mathbb{R}$.

Polytope construction:
Consider a convex lattice polytope $\Delta$ in $M_\mathbb{R}$.

$\leadsto$ define a graded ring (graded by $t_0$) 

$$S_\Delta = \langle t_0^kz^m \rangle_{m \in k\Delta}.$$ 

Grading: $t_0^kz^m \cdot t_0^l z^{m'} = t_0^{k+l}z^{m+m'} \Rightarrow m + m' \in (k + l)\Delta$.

$\leadsto$ projective toric geometry $\mathbb{P}_\Delta = \text{Proj}(S_\Delta)$. 
Cluster varieties

$N^\circ$ is scaling of the lattice $N$.

\[ A = \bigcup T_{N^\circ}/\mu_A \quad \text{and} \quad \mathcal{X} = \bigcup T_M/\mu_{\mathcal{X}} \]

where $\mu_A$ and $\mu_{\mathcal{X}}$ are birational maps between the torus, e.g. $1 + x$, $1 + x^{-1}$.

want: ‘Fan’.
Scattering diagrams

Scattering diagram $\mathcal{D} = \text{collection of walls with finiteness condition}$

wall : $(\mathcal{d}, f_d)$

\[ \cdot \quad \mathcal{d} \subseteq M_{\mathbb{R}} \text{ support of walls - convex rational polyhedral cone of codim 1, contained in } n^{\perp} \in N. \]

\[ \cdot \quad f_d = 1 + \sum c_k z^{k p^*}(n). \]

Example: $A_2$

![Diagram showing the scattering diagram with points at $1 + z^{(0,1)}$, $1 + z^{(-1,0)}$, and $1 + z^{(-1,1)}$.](image-url)
Scattering diagram as fan

\[ f_0 \leadsto \text{wall crossing} \leadsto \text{gluing the } G^2_m \text{'s.} \]
\[ \leadsto \mathcal{A} \text{-cluster variety of type } A_2 \]

Similar construction hold for general setting
Theta functions

To each point \( m \in M^\circ \setminus \{0\} \), associate a theta function \( \vartheta_m \) defined by broken lines:

Example: initial slope \((-1, 0)\):
To each point \( m \in M^\circ \setminus \{0\} \), associate a theta function \( \vartheta_m \) defined by broken lines:

Example: initial slope \((-1, 0)\):

\[
\vartheta_{Q, (-1, 0)} = z^{(-1, 0)} + z^{(-1, 1)}. 
\]
Algebra structure

[Gross-Hacking-Keel-Konsevich] structure constant:

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in L} \alpha_{pq}^r \vartheta_r,$$

where $L = M^\circ$ or $N$ and

$\alpha_{pq}^r$ are expressed in terms of broken lines:

$$\alpha_{pq}^r := \sum_{\left(\gamma^{(1)}, \gamma^{(2)}\right)} c(\gamma^{(1)}) c(\gamma^{(2)}),$$

Example:

$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}.$$
The structure constants endow the vector space generated by theta functions with an algebra structure!
## Toric v.s. Cluster

### Analogy

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<th>Toric</th>
<th>Cluster</th>
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<tr>
<td>toric monomials</td>
<td>theta functions</td>
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<tr>
<td>convex polytope</td>
<td>positive polytope</td>
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</table>
Definition
A closed subset $S \subseteq L_\mathbb{R}$ is positive if
for every $a, b \in \mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$, and $r \in L$ with $\alpha_{pq}^r \neq 0$,
$\Rightarrow r \in (a + b)S$.

Notation: $L = M^\circ$ or $N$, $dS(\mathbb{Z})$ is the cone of $S$ at the ‘d’th-level.

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<tr>
<td>line</td>
<td>broken line</td>
</tr>
<tr>
<td>convex</td>
<td>broken line convex</td>
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</table>
Definition (C-Magee-Nájera Chávez)
A closed subset $S$ is called *broken line convex* if for any $x, y \in S(\mathbb{Q})$, every broken line segment connecting $x$ and $y$ is entirely contained in $S$.

Theorem (C-Magee-Nájera Chávez)
$S$ is positive $\iff S$ is broken line convex.

Idea: The structure constant $\alpha_{pq}^r$ in GHKK were expressed as two broken lines with initial slope $p$ and $q$.

$\star$ [C-Magee-Nájera Chávez] construct the correspondence of those two broken lines with broken line segments with (scaling of) the endpoints $p$ and $q$. 
Compactification

Result:

⇝ get graded ring $R$

⇝ get compactification $\text{Proj}R$

Example:

Type $A_2$:

[Gross-Hacking-Keel-Kontsevich] del Pezzo surface of degree 5
Compactification

Type $B_2$:

[\text{C-Magee}] del Pezzo surface of degree 6

Type $G_2$

\textbf{non-integral} point coming from bending of broken line!
Any evidence?
Why we care?
[Rietsch-Williams] for Grassmannian $\text{Gr}_k(\mathbb{C}^n)$

Newton Okounkov body to $\mathcal{X}$ = Tropicalize the superpotential of $\mathcal{A}$

= positive polytope

Non-integral example from NO body calculation: $\text{Gr}_3(\mathbb{C}^6)$. 
Figure 1: Part of the scattering diagram of $\text{Gr}_3(\mathbb{C}^6)$.

$$\frac{\nu(f)}{2} = \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

Get the non-integral point from broken line convexity!
Newton Okounkov bodies

[... Rietsch-Williams]

\( X = \text{Gr}_{n-k}(\mathbb{C}^n) \), with anticanonical divisor \( D_{ac} = D_1 + \cdots + D_n \).

\( X^\circ = X \setminus D_{ac} \).

Consider ample divisor \( D = r_1D_1 + \cdots r_nD_n \), and the valuation \( \text{val} : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^K \).

Then the NO body for the divisor \( D \) and \( \text{val} \) is

\[
\Delta(D) = \text{ConvexHull} \left( \bigcup_r \frac{1}{r} \text{val}(H^0(X, \mathcal{O}(rD))) \right)
\]
Valuation $\nu_\theta$: $\nu_\theta(\vartheta_p) = p$, where $p \in L$ (set of tropical points)

Intrinsic Newton-Okounkov body

$$\Delta_{BL}^\vartheta(D) = \text{ConvexHull}^{BL} \left( \bigcup_r \frac{1}{r} \nu_\vartheta(H^0(X, O(rD))) \right)$$

Grassmannian: For certain choice of plabic graph (i.e. $\text{val}$, i.e. torus chart),

$$\Delta_{\text{val}}(D) = \text{ConvexHull} (\text{val}(p_j)).$$

[Bossinger-C-Magee-Nájera Chávez] We identify $\text{val}$ with $\nu_\theta$

$$\Delta_{\text{val}}^{BL}(D) = \text{ConvexHull}^{BL} (\text{val}(p_j)),$$

independent of the choice of torus chart.
Batyrev mirror

Batyrev construction: want the polytope $\Delta$ to be reflexive (reflexive means polar dual of $\Delta$ is still a lattice polytope)

Take the points of the primitive generators of the rays of the scattering diagrams

$P = \text{convex hull of these vertices}$

[C-Magee] $P$ is reflexive for type A and $B_2$
Landau Ginzburg mirror

[C-Magee]

<table>
<thead>
<tr>
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<th>Cluster</th>
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<tr>
<td>$X \supset T$ toric Fano, $D = \sum_i D_{n_i}$ toric anticanonical divisor</td>
<td>$(X, D)$ Fano minimal model of cluster variety $U$, $D = \sum_i D_{v_i}$</td>
</tr>
<tr>
<td>$D_{n_i} \sim z^{n_i}$ Landau Ginzburg mirror</td>
<td>$D_{v_i} \sim \vartheta_{v_i}$ Landau Ginzburg mirror</td>
</tr>
<tr>
<td>$W = \sum_i z^{n_i} : T^\vee \to \mathbb{C}$</td>
<td>$W = \sum_i \vartheta_{v_i} : U^\vee \to \mathbb{C}$</td>
</tr>
<tr>
<td>Generic sections of $\mathcal{O}_X(D)$ mildly singular CY hypersurfaces</td>
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<tr>
<td>level sets of $W$</td>
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</tr>
<tr>
<td>want $W$ as sections of some $\mathcal{O}_Y(D')$, $M \subset T^\vee$</td>
<td>want $W$ as sections of some $\mathcal{O}_Y(D')$, $M \subset U^\vee$</td>
</tr>
<tr>
<td>$Y := TV(\text{Newt}(W))$</td>
<td>$\text{Newt}<em>\vartheta(W) := \text{conv}(\vartheta</em>{v_i})$</td>
</tr>
<tr>
<td>Sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror</td>
<td>?</td>
</tr>
</tbody>
</table>
Mirror symmetry

Algebraic geometry ⇔ Symplectic geometry

- (C-Vianna) Mutation of polytopes
- (C-Lin) family Floer mirror
Mutation of polytopes

Cluster mutation of $(\mathcal{X})$-scattering diagram / polytope

‘boring’ as the underlying scheme is not changing
Another mutation

Scattering diagram with monodromy

\[\begin{align*}
1 + z^{e_1} & \quad 1 + z^{-e_2} \\
1 + z^{e_1+e_2} & \quad 1 + z^{e_1+e_2} \\
1 + z^{e_2} & \quad 1 + z^{e_2}
\end{align*}\]

the monodromy: \((1, 0) \mapsto (1, 1), (0, 1) \mapsto (0, 1)\).
Another mutation

Mutation of polytope
[C-Vianna] same as symplectic mutation compactification: singular Lagrangian fibration (almost toric fibration)
Family Floer mirror

Developed by Fakaya, Abouzaid, Tu

Our idea: reinterpret Gross-Hacking-Keel mirror construction in terms of family Floer mirror

Start with \((Y, D)\), where \(Y\) is a smooth rational projective surface, and \(D\) is an anti-canonical cycle of projective lines.

\(\leadsto (B, \Sigma)\), \(B\) affine manifold, \(\Sigma\) cone decomposition of \(B\).

\(\leadsto\) scattering diagram \(\mathcal{D}\) (coming from curve counting)

\(\leadsto\) Theta functions with algebra structure

\(\leadsto\) Take Spec

\(\leadsto\) mirror
<table>
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<tr>
<th><strong>family Floer SYZ</strong></th>
<th><strong>Gross-Hacking-Keel-Siebert mirror</strong></th>
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<tr>
<td>large complex structure limit</td>
<td>toric degeneration</td>
</tr>
<tr>
<td>base of SYZ fibration</td>
<td>dual intersection complex</td>
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<tr>
<td>with complex affine structure</td>
<td>of the toric degeneration</td>
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<td>loci of SYZ fibres bounding holomorphic discs</td>
<td>rays in scattering diagram</td>
</tr>
<tr>
<td>homology of boundary of a holomorphic disc</td>
<td>direction of the ray</td>
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<tr>
<td>exp of generating function</td>
<td></td>
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<tr>
<td>of open Gromov-Witten invariants</td>
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<tr>
<td>of Maslov index zero</td>
<td></td>
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<tr>
<td></td>
<td>slab functions attached to the ray</td>
</tr>
<tr>
<td>coefficients of superpotential =</td>
<td>coefficient of theta functions =</td>
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<tr>
<td>open Gromov-Witten of Maslov index 2 discs</td>
<td>counting of broken lines</td>
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<tr>
<td>isomorphisms of Maurer-Cartan spaces</td>
<td>wall crossing transformation</td>
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<td>induced by pseudo isotopies</td>
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### Construction

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<th>GHKS mirror</th>
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<tr>
<td></td>
<td>$\mathbb{C}[L]$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{C}^n_m$</td>
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<tr>
<td></td>
<td>gluing (wall crossing)</td>
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<td>Tate algebra</td>
<td>$\mathbb{C}[L]$</td>
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<tr>
<td>rational domain</td>
<td>$\mathbb{G}_m^n$</td>
</tr>
<tr>
<td>analytic torus $\mathbb{G}_{\text{an}}^n$</td>
<td>gluing (wall crossing)</td>
</tr>
<tr>
<td>gluing (Wall crossing and GAGA)</td>
<td>gluing (Wall crossing)</td>
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[C-Lin] The family Floer mirror of the hyperKahler rotation of complement of a II* and III* fibres in a rational elliptic surfaces have the compactifications which are the analytification of dP5 and dP6 respectively.
Thank you!