Polytopes, wall crossings, and cluster varieties

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Articles

- Compactifications of cluster varieties and convexity (joint with Magee, Nájera Chávez) arXiv: 1912.13052
- On cluster duality, mirror symmetry and toric degenerations of Grassmannians (joint with Bossinger, Magee and Nájera Chávez), soon!
- Towards Batyrev duality for finite-type cluster varieties (joint with Magee), in preparation
- Algebraic and symplectic viewpoint on compactifications of two-dimensional cluster varieties of finite type (joint with Vianna) arXiv:2008.03265
- Some examples of Family Floer mirror (joint with Lin) soon!

Toric geometry

Fix a lattice $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.
$N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Polytope construction:

Consider a convex lattice polytope Δ in $M_{\mathbb{R}}$.

$\rightsquigarrow$ define a graded ring (graded by t_0)

$$S_{\Delta} = \langle t_0^k z^m \rangle_{m \in k\Delta}.$$

Grading: $t_0^k z^m \cdot t_0^l z^{m'} = t_0^{k+l} z^{m+m'} \Rightarrow m + m' \in (k+l)\Delta$.

$\rightsquigarrow$ projective toric geometry $\mathbb{P}_{\Delta} = \mathrm{Proj}(S_{\Delta})$.

Cluster varieties

N° is scaling of the lattice N.

$$\begin{tikzcd} N^\circ \arrow[d, dashed] & & M \arrow[d, dashed] \\ T_{N^\circ} & & T_M \end{tikzcd}$$

$$\mathcal{A} = \bigcup T_{N^\circ}/\mu_{\mathcal{A}} \qquad \mathcal{X} = \bigcup T_M/\mu_{\mathcal{X}}$$

where $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{X}}$ are birational maps between the torus, e.g. $1+x$. $1+x^{-1}$.

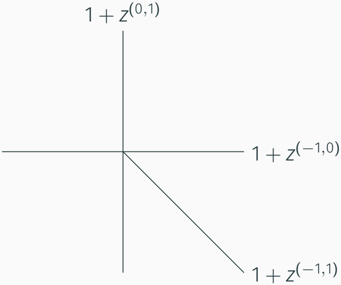
want: ‘Fan’.

Scattering diagrams

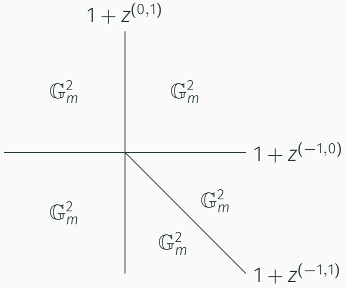
Scattering diagram $\mathfrak{D}$ = collection of walls with finiteness condition
wall : $(\mathfrak{d}, f_{\mathfrak{d}})$

- $\mathfrak{d} \subseteq M_{\mathbb{R}}$ support of walls - convex rational polyhedral cone of codim 1, contained in $n^{\perp} \in N$.
- $f_d = 1 + \sum c_k z^{kp^*(n)}$.

Example: A_2



Scattering diagram as fan



$f_{\mathfrak{d}} \rightsquigarrow$ wall crossing $\rightsquigarrow$ gluing the $\mathbb{G}_m^2$'s.

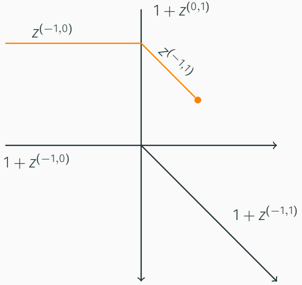
$\rightsquigarrow \mathcal{A}$-cluster variety of type A_2

Similar construction hold for general setting

Theta functions

To each point $m \in M^\circ \setminus \{0\}$, associate a theta function ϑ_m defined by broken lines:

Example: initial slope $(-1, 0)$:



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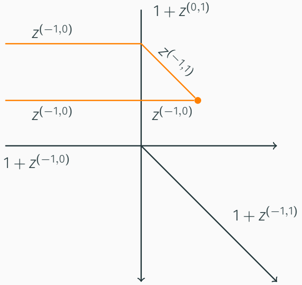


$$\vartheta_{Q,(-1,0)} = z^{(-1,0)} + z^{(-1,1)}.$$

Algebra structure

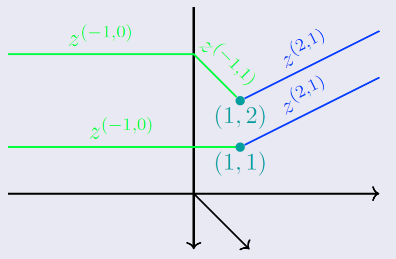
[Gross-Hacking-Keel-Konsevich] structure constant:

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in L} \alpha_{pq}^r \vartheta_r,$$

where $L = M^\circ$ or N and

α_{pq}^r are expressed in terms of broken lines:

$$\alpha_{pq}^r := \sum_{\substack{(\gamma^{(1)},\gamma^{(2)}) \\ I(\gamma^{(1)})=p,\ I(\gamma^{(2)})=q \\ \gamma^{(1)}(0)=\gamma^{(2)}(0)=r \\ F(\gamma^{(1)})+F(\gamma^{(2)})=r}} c(\gamma^{(1)})\, c(\gamma^{(2)}),$$



Example:

$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}.$$

♠ The structure constants endow the vector space generated by theta functions with an algebra structure!

Toric v.s. Cluster

Analogy

Toric	Cluster
fan	scattering diagram
toric monomials	theta functions
convex polytope	positive polytope

Positive polytope

Definition
A closed subset $S \subseteq L_{\mathbb{R}}$ is *positive* if
for every $a, b \in \mathbb{Z}_{\geq 0}$, $p \in aS(\mathbb{Z})$, $q \in bS(\mathbb{Z})$, and $r \in L$ with $\alpha^r_{pq} \neq 0$,

$\Rightarrow r \in (a+b)S$.

Notation: $L = M^\circ$ or N, $dS(\mathbb{Z})$ is the cone of S at the 'd'th-level.

Toric	Cluster
fan	scattering diagram
toric monomials	theta functions
convex polytope	positive polytope
line	broken line
convex	broken line convex

Broken line convex

Definition (C-Magee-Nájera Chávez)
A closed subset S is called *broken line convex* if for any $x, y \in S(\mathbb{Q})$, every broken line segment connecting x and y is entirely contained in S.

Theorem (C-Magee-Nájera Chávez)
S is positive $\Leftrightarrow$ S is broken line convex.

Idea: The structure constant α_{pq}^{r} in GHKK were expressed as two broken lines with initial slope p and q.

$\star$ [C-Magee-Nájera Chávez] construct the correspondence of those two broken lines with broken line segments with (scaling of) the endpoints p and q.

Compactification

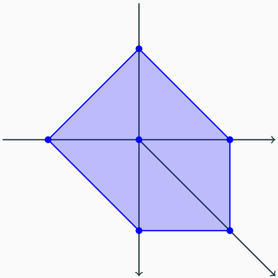
Result:

$\rightsquigarrow$ get graded ring R

$\rightsquigarrow$ get compactification $\mathrm{Proj} R$

Example:

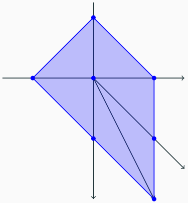
Type A_2:



[Gross-Hacking-Keel-Kontsevich]del Pezzo surface of degree 5

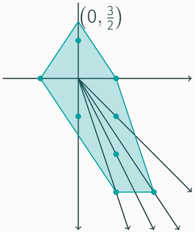
Compactification

Type B_2:



[C-Magee] del Pezzo surface of degree 6

Type G_2



non-integral point coming from bending of broken line!

Any evidence?
Why we care?

Grassmannian

[Rietsch-Williams] for Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$

$$\begin{array}{c}\text{Newton Okounkov body}\\ \text{to } \mathcal{X}\end{array} = \begin{array}{c}\text{Tropicalize the superpotential}\\ \text{of } \mathcal{A}\end{array}$$

$$= \text{positive polytope}$$

Non-integral example from NO body calculation: $\mathrm{Gr}_3(\mathbb{C}^6)$.

[Bossinger-C-Magee-Nájera Chávez]

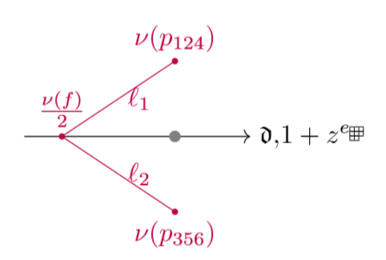


Figure 1: Part of the scattering diagram of $\mathrm{Gr}_3(\mathbb{C}^6)$.

$$\frac{\nu(f)}{2} = \left(\frac{1}{2}, 1, \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$$

Get the non-integral point from broken line convexity!

Newton Okounkov bodies

[..., Rietsch-Williams]

$\mathbb{X} = \mathrm{Gr}_{n-k}(\mathbb{C}^n)$, with anticanonical divisor $D_{\mathrm{ac}} = D_1 + \cdots + D_n$.
$\mathbb{X}^\circ = \mathbb{X} \setminus D_{\mathrm{ac}}$.
Consider ample divisor $D = r_1 D_1 + \ldots r_n D_n$, and
the valuation $\mathrm{val} : \mathbb{C}(\mathbb{X}) \setminus \{0\} \to \mathbb{Z}^K$.

Then the NO body for the divisor D and val is

$$\Delta(D) = \overline{\mathrm{ConvexHull}\left(\bigcup_r \frac{1}{r}\mathrm{val}(H^0(\mathbb{X}, \mathcal{O}(rD)))\right)}$$

NO bodies for cluster varieties

Valuation ν_θ: $\nu_\theta(\vartheta_p) = p$, where $p \in L$ (set of tropical points)

Intrinsic Newton-Okounkov body

$$\Delta_\vartheta^{\mathrm{BL}}(D) = \overline{\mathrm{ConvexHull}^{\mathrm{BL}}\left(\bigcup_r \frac{1}{r}\nu_\vartheta(H^0(\mathbb{X}, \mathcal{O}(rD)))\right)}$$

Grassmannian: For certain choice of plabic graph (i.e. val, i.e. torus chart),

$$\Delta_{\mathrm{val}}(D) = \mathrm{ConvexHull}\left(\mathrm{val}(p_J)\right).$$

[Bossinger-C-Magee-Nájera Chávez] We identify val with ν_θ

$$\Delta_{\mathrm{val}}^{\mathrm{BL}}(D) = \mathrm{ConvexHull}^{\mathrm{BL}}\left(\mathrm{val}(p_J)\right),$$

independent of the choice of torus chart.

Batyrev mirror

Batyrev construction: want the polytope Δ to be reflexive
(reflexive means polar dual of Δ is still a lattice polytope)

Take the points of the primitive generators of the rays of the scattering diagrams
P = convex hull of these vertices

[C-Magee] P is reflexive for type A and B_2

Landau Ginzburg mirror

[C-Magee]

Toric	Cluster
$X \supset T$ toric Fano, $D = \sum_i D_{n_i}$ toric anticanonical divisor	(X, D) Fano minimal model of cluster variety U, $D = \sum_i D_{v_i}$
$D_{n_i} \rightsquigarrow z^{n_i}$ Laudau Ginzburg mirror $W = \sum_i z^{n_i} : T^\vee \to \mathbb{C}$	$D_{v_i} \rightsquigarrow \vartheta_{v_i}$ Laudau Ginzburg mirror $W = \sum_i \vartheta_{v_i} : U^\vee \to \mathbb{C}$
Generic sections of $\mathcal{O}_X(D)$ mildly singular CY hypersurfaces	Generic sections of $\mathcal{O}_X(D)$ mildly singular CY hypersurfaces
level sets of W	level sets of W
want W as sections of some $\mathcal{O}_Y(D')$, $M \subset T^\vee$	want W as sections of some $\mathcal{O}_Y(D')$, $M \subset U^\vee$
$Y := \mathrm{TV}(\mathrm{Newt}(W))$	$\mathrm{Newt}_\vartheta(W) := \mathrm{conv}(v_i)$
Sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_Y(D')$ are mirror	?

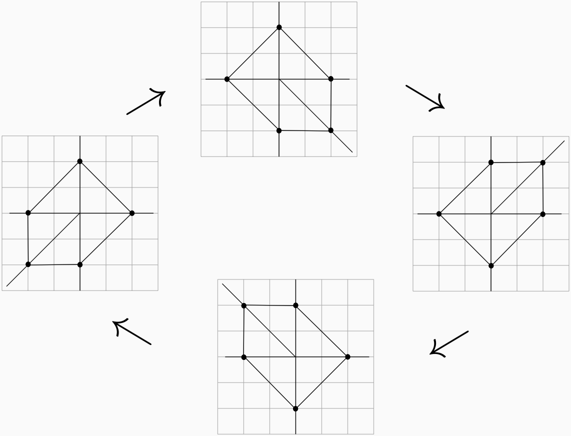
Mirror symmetry

$$\text{Algebraic geometry} \iff \text{Symplectic geometry}$$

- (C-Vianna) Mutation of polytopes
- (C-Lin) family Floer mirror

Mutation of polytopes

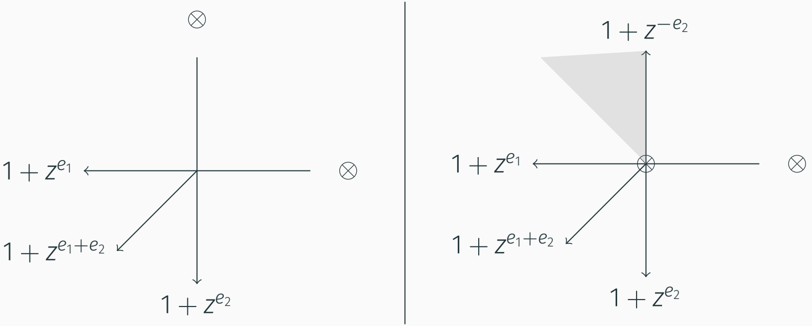
Cluster mutation of ($\mathcal{X}$-)scattering diagram / polytope



'boring' as the underlying scheme is not changing

Another mutation

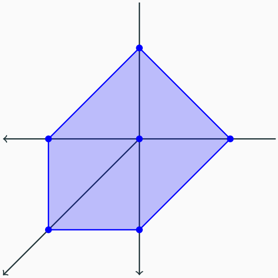
Scattering diagram with monodrony

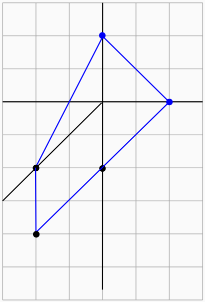


the monodromy: $(1,0) \mapsto (1,1)$, $(0,1) \mapsto (0,1)$.

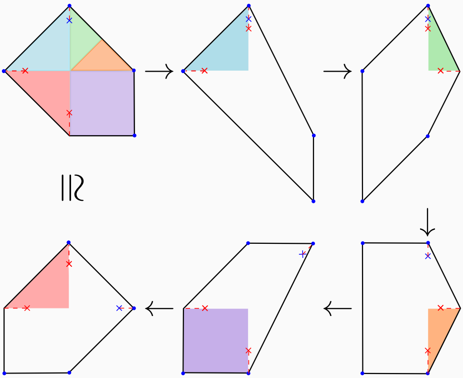
Another mutation

Mutation of polytope





Mutation cycle



[C-Vianna] same as symplectic mutation compactification: singular Lagrangian fibration (almost toric fibration)

Family Floer mirror

Developed by Fakaya, Abouzaid, Tu

Our idea: reinterpret Gross-Hacking-Keel mirror construction in terms of family Floer mirror

Start with (Y, D), where Y is a smooth rational projective surface, and D is an anti-canonical cycle of projective lines.

$\rightsquigarrow$ (B, Σ), B affine manifold, Σ cone decomposition of B.

$\rightsquigarrow$ scattering diagram $\mathfrak{D}$ (coming from curve counting)

$\rightsquigarrow$ Theta functions with algebra structure

$\rightsquigarrow$ Take Spec

$\rightsquigarrow$ mirror

SG v.s. AG

family Floer SYZ	**Gross-Hacking-Keel-Siebert mirror**
large complex structure limit	toric degeneration
base of SYZ fibration with complex affine structure	dual intersection complex of the toric degeneration
loci of SYZ fibres bounding holomorphic discs	rays in scattering diagram
homology of boundary of a holomorphic disc	direction of the ray
exp of generating function of open Gromov-Witten invariants of Maslov index zero	slab functions attached to the ray
coefficients of superpotential = open Gromov-Witten of Maslov index 2 discs	coefficient of theta functions = counting of broken lines
isomorphisms of Maurer-Cartan spaces induced by pseudo isotopies	wall crossing transformation

Construction

family Floer mirror	GHKS mirror
	$\mathbb{C}[L]$
	$\mathbb{G}_m^n$
	gluing (wall crossing)

Construction

family Floer mirror	GHKS mirror
Tate algebra	$\mathbb{C}[L]$
rational domain analytic torus $\mathbb{G}^n_{\mathrm{an}}$	$\mathbb{G}^n_m$
gluing (Wall crossing and GAGA)	gluing (wall crossing)

[C-Lin] The family Floer mirror of the hyperKahler rotation of complement of a II^* and III^* fibres in a rational elliptic surfaces have the compactifications which are the analytification of dP5 and dP6 respectively.

Thank you!