MAT 280: PROBLEM SET 2

DUE TO FRIDAY OCT 8 AT 9:00PM

Abstract. This is the second problem set for the graduate course Contact and Symplectic Topology in the Fall Quarter 2021. It was posted online on Friday Oct 1 and is due Friday Oct 8 at 9:00pm via online submission.

Task and Grade: Solve one of the five problems Problem 1 through Problem 5 below.[1]The maximum possible grade is 100 points. Despite the task being one problem, I strongly encourage you to work on the five problems.

Instructions: It is good to consult with other students and collaborate when working on the problems. You should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Problem 1. Let $(\mathbb{R}^3, \xi_{st})$ be the standard contact structure $\xi_{st} = \ker\{dz - ydx\}$, as depicted in Figure 1, where $(x, y, z) \in \mathbb{R}^3$ are Cartesian coordinates.

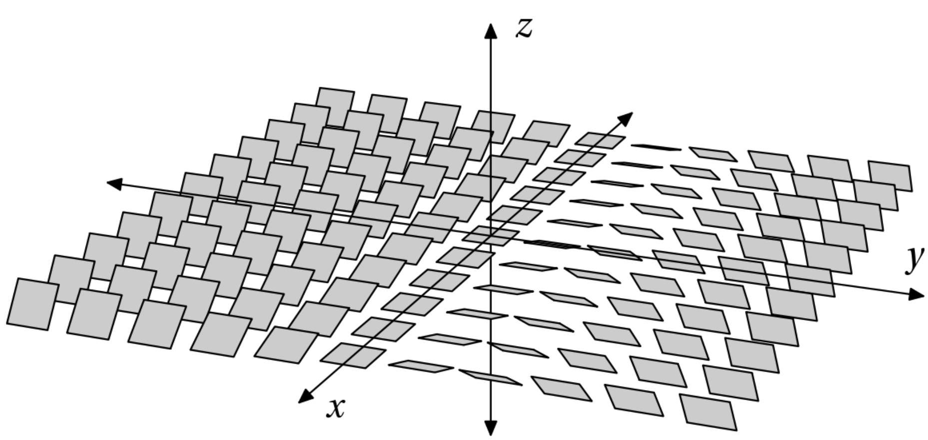

Figure 1. A depiction of $\xi_{st} = \ker\{dz - ydx\}$. Note that the contact structure is z-invariant, and thus it suffices to depict it in $\{z = 0\}$.

(1) For each of the following subsets $S \subseteq \mathbb{R}^3$, decide whether they are Legendrian, transverse, or neither:

(a) The z-axis $S = \{x = y = 0\}$, (b) The y-axis $S = \{x = z = 0\}$,

[1]There is also a practice problem if you want to think about smooth knots, but it is not graded.

(c) The x-axis $S = \{z = y = 0\}$, (d) $S = \{(t^2, 3t/2, t^3) : t \in [-1, 1]\}$,

(e) The line $S = \{z = -283.1, x = c\}$, $c \in \mathbb{R} \setminus \{0\}$ a fixed constant.

(f) The line $S = \{z = 4.3, y = C\}$, $C \in \mathbb{R} \setminus \{0\}$ a fixed constant.

(2) Let $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^2_{x,z}$ be given by

$$\Gamma_1 = \{x^3 = z^2\}, \qquad \Gamma_2 = \{(x - z)(x + z) = 0\}.$$

Draw Legendrian lifts $\Lambda_{\Gamma_1}, \Lambda_{\Gamma_2} \subseteq (\mathbb{R}^3, \xi_{st})$ for each of these sets.[2]

(3) Draw the images $\Pi(\Lambda_{\Gamma_1})$ and $\Pi(\Lambda_{\Gamma_2})$ in $\mathbb{R}^2_{x,y}$, where $\Pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the projection $\Pi(x, y, z) = (x, y)$. Do the images $\Pi(\Lambda_{\Gamma_1})$ and $\Pi(\Lambda_{\Gamma_2})$ in $\mathbb{R}^2_{x,y}$ recover the Legendrians $\Lambda_{\Gamma_1}, \Lambda_{\Gamma_2} \subseteq (\mathbb{R}^3, \xi_{st})$?

(4) Does there exist an embedded Legendrian curve $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ whose projection $\pi(\Lambda) \subseteq \mathbb{R}^2_{x,z}$ is the set $\Gamma = \{x = z^2\}$?

(5) Does there exist an embedded Legendrian curve $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ whose projection $\pi(\Lambda) \subseteq \mathbb{R}^2_{x,z}$ is the set $\Gamma = \{x^5 = z^2\}$?

(Here $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the Legendrian front projection $\pi(x, y, z) = (x, z)$.)

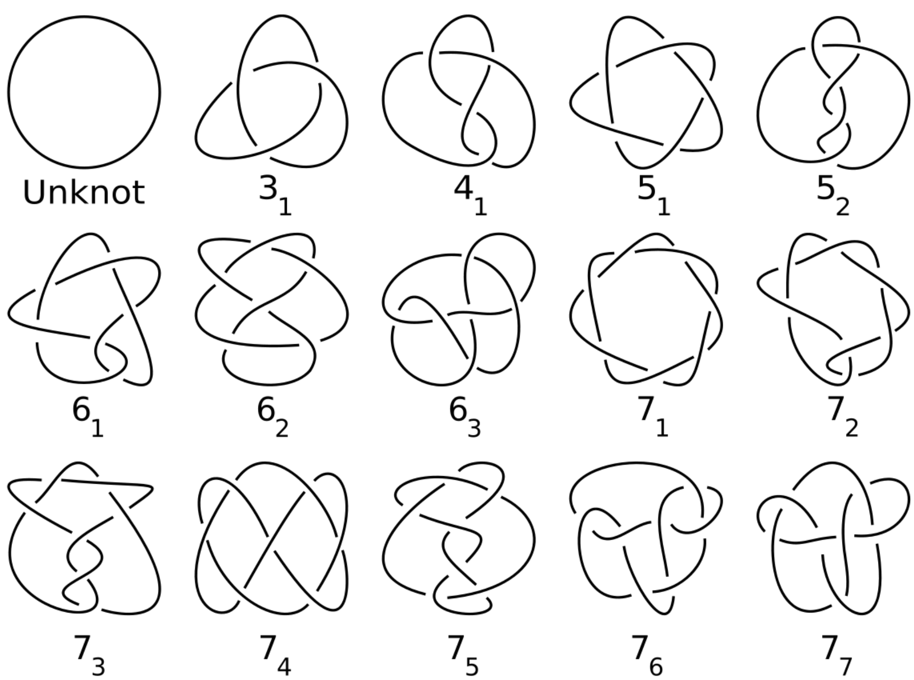

FIGURE 2. Table with a few smooth knots whose knot diagrams have a small number of crossings. This is used in Problems 2 and 3.

[2] A Legendrian lift of $\gamma \subseteq \mathbb{R}^2$ is an embedded Legendrian curve $\Lambda_\gamma \subseteq (\mathbb{R}^3, \xi_{st})$ that front projects to γ, under the front projection π, i.e. $\pi(\Lambda_\gamma) = \gamma$.

Problem 2. Let $(\mathbb{R}^3, \xi_{st})$ be the standard contact structure $\xi_{st} = \ker\{dz - ydx\}$. Given a Legendrian front $\Gamma \subseteq \mathbb{R}^2_{x,z}$, consider a Legendrian lift $\Lambda_\Gamma \subseteq (\mathbb{R}^3, \xi_{st})$. For each of the Legendrian fronts $\Gamma \subseteq \mathbb{R}^2_{x,z}$ in Figure 3, except Figure 3.(3), determine the smooth type of the Legendrian knot $\Lambda_\Gamma \subseteq (\mathbb{R}^3, \xi_{st})$, i.e. name the smooth type of the knot according to the knot tables and prove your claim.

A table with smooth knots is provided in Figure 2. Note that knot tables do not take into account mirrors. Also, Figure 3.(3) is a 2-component link, which link is it?

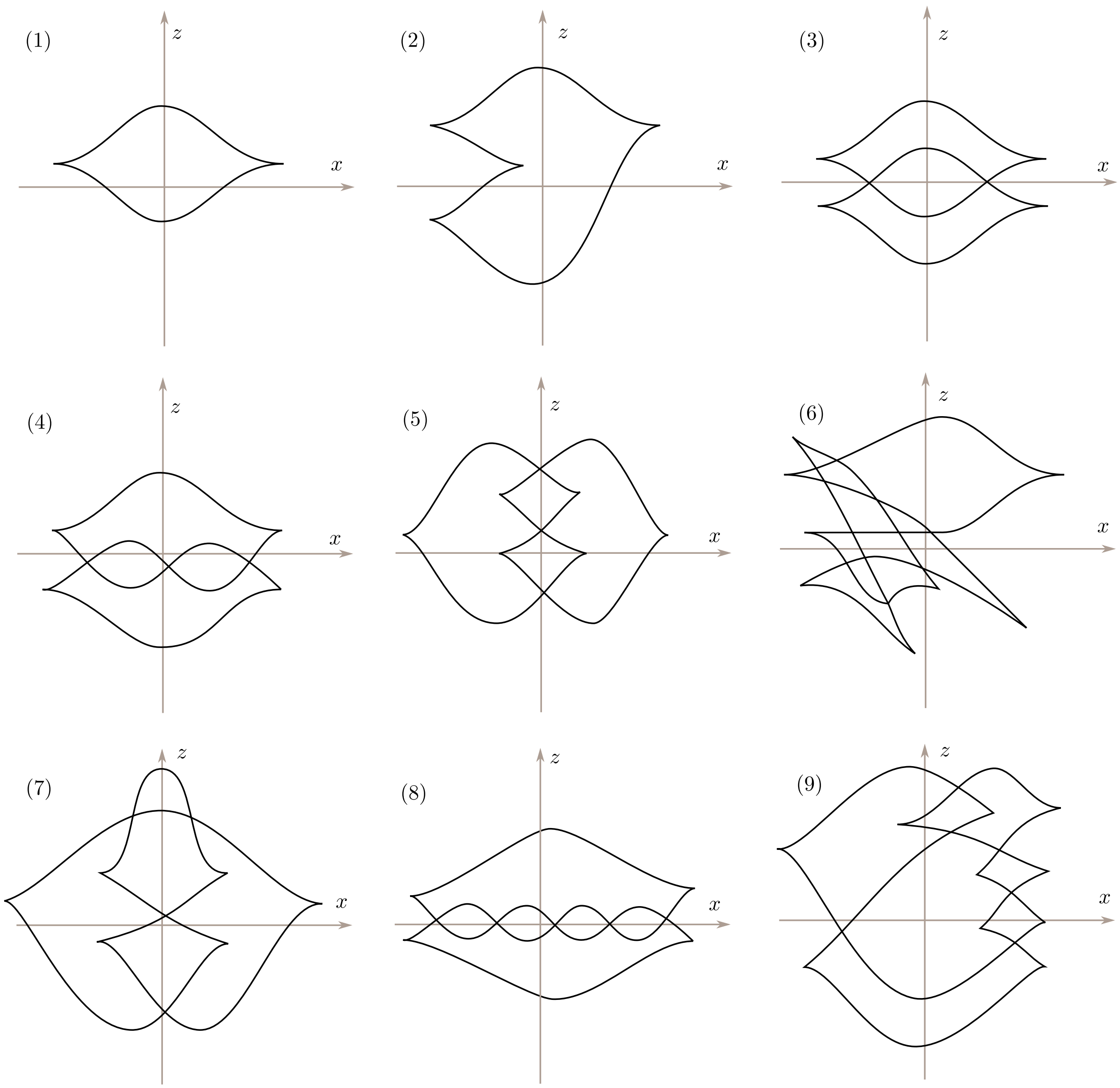

FIGURE 3. Nine Legendrian fronts. All cusps are semi-cuspidal, i.e. locally parametrized by $\gamma(t) = (t^2, t^3)$.

Problem 3. (Realizing Smooth Knots as Legendrians) Solve the following two parts:

(i) For each of the smooth knots $K \subseteq \mathbb{R}^3$ depicted in Figure 2, draw at least one Legendrian knot $\Lambda(K) \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ such that the smooth isotopy class of the Legendrian knot $\Lambda(K)$ is the one given by the smooth knot K.

Hint: You may describe $\Lambda(K)$ through its front projection $\pi(\Lambda(K))$.

(ii) Consider a trefoil knot $K_{3_1} \subseteq \mathbb{R}^3$ whose smooth knot diagram when projected onto the (x, z)-plane, with the information of the over- and under-crossings, is the diagram for the trefoil knot (3_1) in Figure 2. (First row, second column). Construct a Legendrian trefoil knot $\Lambda_{3_1} \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ which is C^0-close to K_{3_1}.

Remark: It is likely that the Legendrian representative of the trefoil that you drew in (i) will not be C^0-close. Try to understand if that is the case.

Problem 4. (A taste of Generating Families) Let us consider the family of functions $f_x : \mathbb{R} \longrightarrow \mathbb{R}$ defined $f_x(\tau) = \tau^3 + (|x|^2 - 1)\tau$, where $\tau \in \mathbb{R}$ and $x \in \mathbb{R}$ is thought of as a parameter.

(i) Consider the 2-plane $\mathbb{R}^2_{x,z}$ and draw the set of *critical values of* f_τ, i.e.

$$S(f_x) = \{(x, z) : z \text{ is a critical value of } f_x\} = \{(x, z) : \exists \zeta \text{ s.t. } df_x(\zeta) = 0, z = f(\zeta)\} \subseteq \mathbb{R}^2.$$

(ii) If you have drawn (i) correctly, S will be a front projection of a Legendrian knot $\Lambda_S \subseteq (\mathbb{R}^3, \xi_{\text{st}})$. What is the smooth type of this knot?

(iii) Find a family of functions $f_x : \mathbb{R} \longrightarrow \mathbb{R}$, where $x \in \mathbb{R}$ is a parameter, such that $S(f_x)$ is the front drawn in Figure 3.(4).

(iv) Show that there is no family of functions $f_x : \mathbb{R} \longrightarrow \mathbb{R}$, where $x \in \mathbb{R}^m$ is again a parameter, $m \in \mathbb{N}$, such that $S(f_x)$ is the front drawn in Figure 3.(2).

For Context: In general, a Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ which can be described as $\Lambda = \Lambda_{S(f_x)}$ for some family of functions f_x, $x \in \mathbb{R}^m$, for some $m \in \mathbb{N}$, is said to admit a generating family f_x. For instance, (i) and (iii) above show that certain Legendrian knots admit generating families, where (iv) shows that not all Legendrian knots admits generating families.

Generating families originate in geometric optics, which are modeled with contact geometry. (The Huygens principle is precisely a statement about contactomorphisms.) This is one physical reason why (wave)fronts are visible. For instance, the cusp that you see in a coffee mug when you direct light to it from above is Lagrangian caustic, which is closely related to a Legendrian front. In modern contact topology, generating functions are becoming encompassed by the theory of microlocal sheaves.

Problem 5. (Overtwisted Contact Structures) Let $(r,\theta,z)\in\mathbb{R}^3$ be cylindrical coordiantes in $\mathbb{R}^3$. Consider the 1-form

$$\alpha_{ot}=\cos(r)dz+r\sin(r)d\theta,\quad r\in\mathbb{R}_{\geq 0},\theta\in[0,2\pi).$$

(i) Show that $\ker\alpha_{ot}$ is a contact structure and qualitatively draw it.

(ii) Show that the subset $C=\{z=0,r=\pi\}$ is a Legendrian subspace and $C\cong\mathbb{S}^1$ is diffeomorphic to a 1-sphere $\mathbb{S}^1$.

(iii) Let $\mathcal{O}p\,(0)=\{r^2+z^2\leq\varepsilon\}$ be a small round ball around the origin, $\varepsilon\in\mathbb{R}^+0$ arbitrarily small but fixed. Show that there exist coordinates for $\mathcal{O}p\,(0)$ such that $(\mathcal{O}p\,(0),\ker(\alpha_{ot}))$ is contactomorphic to $(\mathbb{R}^3,\ker(\alpha_{st}))$, where $\alpha_{st}=dz-ydx$, with $(x,y,z)\in\mathbb{R}^3$ Cartesian coordinates.

(iv) In $(\mathbb{R}^3,\xi_{\mathrm{st}})$, the global front projection map $\pi:(\mathbb{R}^3,\xi_{\mathrm{st}})\longrightarrow\mathbb{R}^2$, $\pi(x,y,z)=(x,z)$ has Legendrian fibers. Show that, in contrast, it does have Legendrian fibers for $(\mathbb{R}^3,\xi_{ot})$. Does $(\mathbb{R}^3,\xi_{ot})$ admit any front projection onto $\mathbb{R}^2$?[3]

(v) (Optional and Hard) Show that there exists no diffeomorphism $\phi:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ such that $\ker(\phi^*(\alpha_{ot}))=\ker(\alpha_{st})$, i.e. $\ker\alpha_{ot}$ is not contactomorphic to $\ker\alpha_{st}$.

The contact structure $\ker\alpha_{ot}$ is known as an *overtwisted* contact structure, and it is historically the first example of a contact structure on $\mathbb{R}^3$ which is not contactomorphic to $(\mathbb{R}^3,\xi_{\mathrm{st}})$. This was proven by D. Bennequin in 1983.[4]

Practice problem. This is a practice problem on smooth knots, there is no contact geometry involved. Consider the table in Figure 2, where some smooth knots with few crossings are depicted

.

(i) Show that the unknot in Figure 2 is not smoothly isotopic to the trefoil 3_1.

(ii) Show that the unknot in Figure 2 is not smoothly isotopic to 6_1, 7_4 or 7_7.

(iii) Consider the knot $m(3_1)$, called the mirror of the trefoil knot, whose knot diagram is given by the knot diagram of the trefoil in Figure 2, but where each crossing is switched[5] Show that 3_1 is not smoothly isotopic to $m(3_1)$.

This shows that there are at least three distinct smooth isotopy classes of knots $K\subseteq\mathbb{R}^3$, the unknot, the trefoil 3_1 and its mirror $m(3_1)$. (In order to distinguish many more, one needs to construct systematic invariants, such as the Alexander or Jones polynomials.)

[3] That is, does there exist a global map $\rho:(\mathbb{R}^3,\xi_{ot})\longrightarrow\mathbb{R}^2$ whose fibers are all Legendrian?

[4] En Français, on leur appelle des "structures de contact *vrillées*": d'après une vrille en botanique, la pièce foliaire correspondant à un organe spécialisé permettant à certaines plantes grimpantes de s'accrocher à des supports divers.

[5] Any undercrossing becomes an overcrossing, and viceversa.