MAT 280: PROBLEM SET 4

DUE TO FRIDAY OCT 22 AT 9:00PM

Abstract. This is the fourth problem set for the graduate course Contact and Symplectic Topology in the Fall Quarter 2021. It was posted online on Friday Oct 15 and is due Friday Oct 22 at 9:00pm via online submission.

Task and Grade: Solve one of the five problems Problem 1 through Problem 5 below.[1]The maximum possible grade is 100 points. Despite the task being one problem, I strongly encourage you to work on the five problems. Problem 0 is for fun: studying smooth functions from $\mathbb{R}$ to $\mathbb{R}$ can be cool.

Instructions: It is good to consult with other students and collaborate when working on the problems. You should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. By convention, all fronts are oriented by choose the highest point and adding an arrow to the right.

Notation: In class, we have adopted the notation that generating families are $f_x : \mathbb{R}^q \longrightarrow \mathbb{R}$, with $x \in \mathbb{R}$ a real parameter, and $\tau = (\tau_1, \ldots, \tau_q) \in \mathbb{R}^q$ the (multi)variable for the domain $\mathbb{R}^q$. The coordinate variable for the target $\mathbb{R}$ is $z \in \mathbb{R}$. Recall that the front associated to f_x is

$$\Sigma(f_x) := \{(x, f_x(\tau_0)) : \tau_0 \in \mathbb{R}^q, \quad df_x(\tau_0) = 0\} \subseteq \mathbb{R}^2_{x,z},$$

i.e. the pairs (x_0, z_0) where z_0 is a critical value of the function $f_{x_0} : \mathbb{R}^q \longrightarrow \mathbb{R}$.

Problem 0. A smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be non-degenerate (a.k.a. Morse) if its critical points $\{x \in \mathbb{R} : df(x) = 0\}$ are isolated minima and isolated maxima, i.e. if the Hessian at each critical point is a non-degenerate matrix. Two smooth functions $f_1, f_2 : \mathbb{R}_x \longrightarrow \mathbb{R}_y$ will be said to be equivalent if they can be transformed one into the other by a smooth change of the independent x and dependent variables y, i.e. diffeomorphisms of the domain and target.

Find the number $K(n)$ of pairwise non-equivalent non-degenerate functions of one variable having n non-degenerate critical points with pairwise different critical values, supposing that at infinity the function behaves like the linear function x if n is even, and like the function x^2 for n odd.

(*Hint*: What is the Taylor series of $\sec(t) + \tan(t)$?)

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[1]There is also a practice problem if you want to think about smooth knots, but it is not graded.

Problem 1. Solve the following parts:

(i) Show that the Legendrian knots whose fronts are drawn in Figure 1.(1) and (2) admit a generating family with $q = 1$, i.e. a generating family $f_x : \mathbb{R} \longrightarrow \mathbb{R}$ with $x \in \mathbb{R}$ as a parameter.

(ii) Do the Legendrian knots whose fronts are drawn in Figure 1.(1) and (2) admit a generating family with $q \geq 2$.

(iii) Show that a stabilized Legendrian knot, such as the one depicted in Figure 1.(3), does *not* admit a generating family for any $q \in \mathbb{N}$.

(iv) Show that the Legendrian knot in Figure 1.(4) does not admit a generating family with $q = 1$, but admits a generating family with $q = 2$.

(v) For any $m \in \mathbb{N}$, construct a Legendrian knot which does not admit a generating family for $q = m$, but admits a generating family for $q \geq m+1$. (Prove it.)

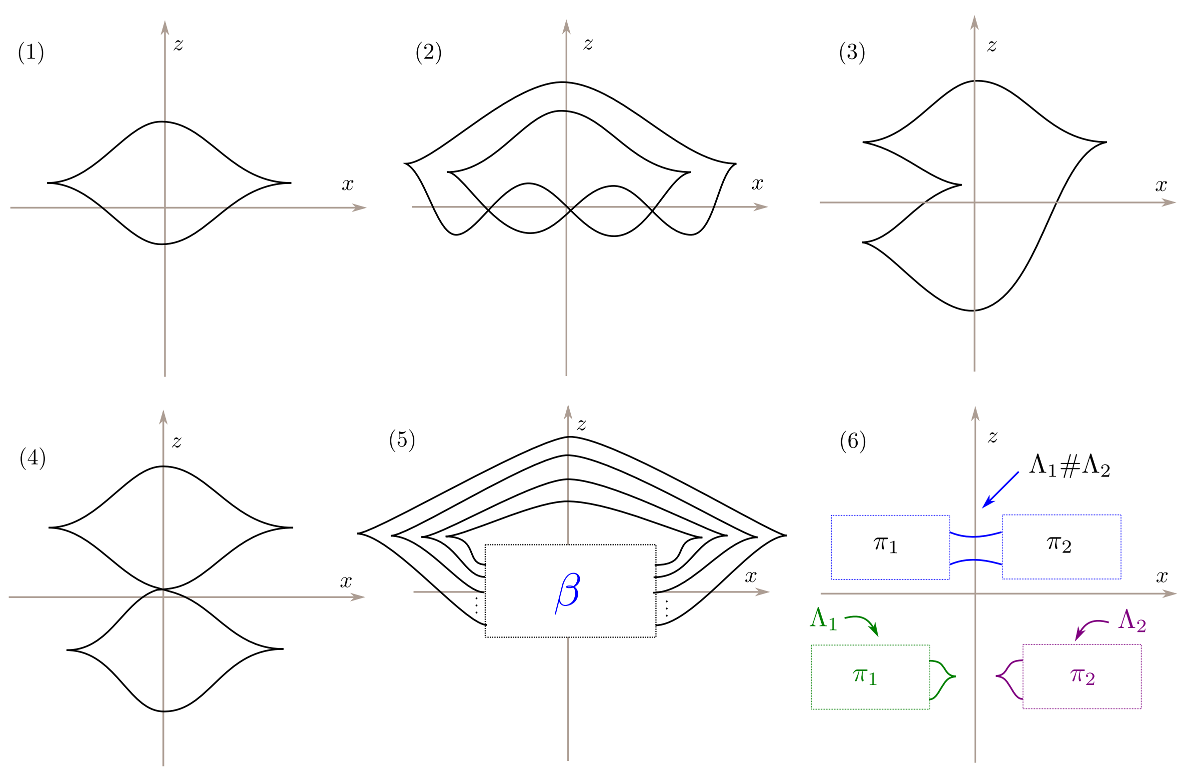

FIGURE 1. Legendrian fronts for Problem Set 4.

Problem 2. Solve the following parts:

(i) Let $n \in \mathbb{N}$ be a natural number, and $[\beta] \in \mathrm{Br}_n^+$ an n-stranded positive braid with braid word $\beta = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_\ell}$, where σ_j, $1 \leq j \leq n-1$ are the Artin generators of Br_n^+, i.e. one positive crossing between the strands j and $j+1$. Consider the Legendrian link $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ whose front is depicted in Figure 1.(5). Show that $\Lambda(\beta) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ admits a generating family.

(ii) Let Λ_1, Λ_2 be the Legendrian knots in Figure 1.(5) whose fronts are depicted as π_1 with a right cusp, and π_2 with a keft cusp, in the drawing. Suppose that each Λ_1 and Λ_2 admit a generating family and consider their Legendrian connected sum $\Lambda_1\#\Lambda_2$, a front of which is depicted in Figure 1.(5). Does $\Lambda_1\#\Lambda_2$ admits a generating family?

Problem 3. Consider the two Legendrian knots $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ in Figure 2, whose fronts are denoted by $\pi_0, \pi_1 \subseteq \mathbb{R}^2_{x,z}$. Let us denote $\Pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ for the projection $\Pi(\tau, x, z) = (x, z)$.

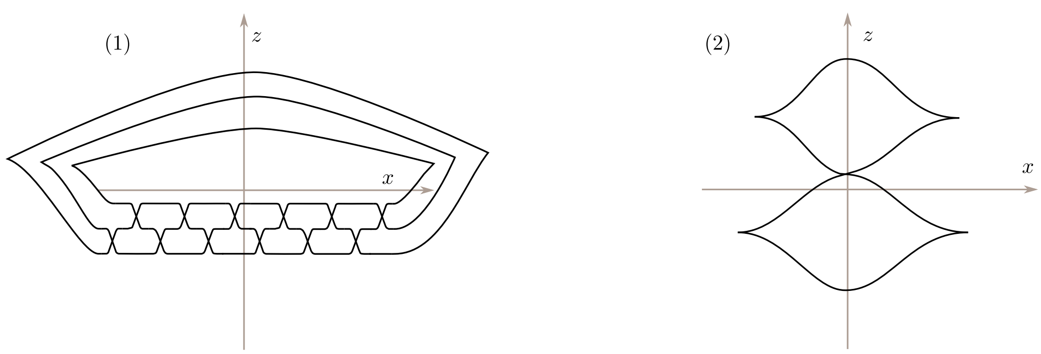

FIGURE 2. Two fronts representing the Legendrian torus link $T(3,6)$, denoted Λ_0, on the left, and the standard Legendrian unknot Λ_1 with a Reidemeister I move, on the right.

(i) Show that both Λ_0 and Λ_1 admit generating families $f_x^{(0)} : \mathbb{R}_\tau \longrightarrow \mathbb{R}_z$, where $\tau \in \mathbb{R}$, and $f_x^{(1)} : \mathbb{R}^2_\tau \longrightarrow \mathbb{R}_z$, where $\tau = (\tau_1, \tau_2)$.

(ii) For each bounded connected component $C \in \mathbb{R}^2_{x,z} \setminus \pi_0$ of the complement of the front π_0, let $B_\varepsilon(C) \subseteq C$ be a ball of radius ε entirely contained in C. (Thus $\varepsilon \in \mathbb{R}^+$ is chosen small enough.). For each bounded connected component $C \in \mathbb{R}^2_{x,z} \setminus \pi_j$, describe the space given by the intersection

$$X(C) := \Pi^{-1}(B_\varepsilon(C)) \cap \{(\tau, x, z) : f_x^{(0)}(\tau) \leq z\} \subseteq \mathbb{R}^3.$$

(In particular, are the components of $X(C)$ always contractible?)

(iii) Label each bounded connected component $C \in \mathbb{R}^2_{x,z} \setminus \pi_0$ of the complement of the front π_0 with the number of connected components of $X(C)$.

(iv) Repeat parts (ii) and (iii) with the front π_1 and the generating family $f_x^{(1)}$.

Problem 4. Let $\pi_s \subseteq \mathbb{R}^2_{x,z}$ be the family of fronts in a Reidemeister I move, $0 \leq s \leq 1$.

(i) Draw the surface in $\mathbb{R}^3 = \mathbb{R}^2_{x,z} \times \mathbb{R}_s$ obtained by the union of those fronts.[2]

(ii) Consider the set $\mathbb{R}^3 \cong \{x^4 + ax^2 + bx + c\}$, $a, b, c \in \mathbb{R}^3$, of polynomials of degree 4 whose roots sum to zero. Draw the subset $S \subseteq \mathbb{R}^3$ of polynomials that have a multiple root.

(iii) Consider the smooth map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$f(x_1, x_2, x_3) = (x_1^4 + x_1^2 x_2 + x_1 x_3, x_2, x_3).$$

Show that the set of critical points of this map f is a smooth surface $\mathbb{S} \subseteq \mathbb{R}^3$. Draw the subset $S \subseteq \mathbb{R}^3$ of critical values $S = f(\mathbb{S})$.

(iv) (Optional) Consider the curve $\gamma(t) = (t^2, t^3, t^4) \in \mathbb{R}^3$ in 3-space. Draw the set $S \subseteq \mathbb{R}^3$ given as the union of all the tangent lines of the curve $\gamma \subseteq \mathbb{R}^3$.

Problem 5. Let $\Gamma_0 = \{(q,p) : p = q^2\} \subseteq \mathbb{R}^2_{(q,p)}$ be the parabola curve in the plane. This is considered as an initial curve at $t = 0$ which is to propagate in the plane $\mathbb{R}^2_{(q,p)}$, as a source of light propagates in optics, or a wavefront of water propagates in a liquid surface. The curve Γ_0 propagates as follows: the position of the curve Γ_t at time t is an equidistant curve of Γ_0. To obtain this equidistant curve, one considers the set of (inwards pointing, i.e. upwards for Γ_0) normal vectors along Γ_0. Then the equidistant curve Γ_t is formed by the points in the plane at distance t in the forward direction from the points of the initial curve *along each normal*. Note that for small enough time t, the curves are Γ_t, but eventually Γ_t acquires singularities.

(i) Draw the family of curves Γ_t, $t \geq 0$ until Γ_t has at least three singular points.

(ii) Explain how the family of curves Γ_t are related to Legendrian curves.

(iii) Describe a relation between the family of curves Γ_t and the caustic of light, seen there as a cusp, in Figure 3.

FIGURE 3. The behaviour of light when reflecting on a coffee mug.

[2]This is a singular surface known as the *swallowtail*. It inspired the painting "The Swallow's Tail" by S. Dalí. It is worth googling "swallowtail singularity".