University of California Davis **Name (Print):** ____________________
Differential Topology MAT-239 **Student ID (Print):** ____________________

Midterm Examination II **Nov 25 2024**
Time Limit: 50min

This examination document contains 6 pages, including this cover page, and 4 problems.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) Consider $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z), $r \in \mathbb{R}_{\geq 0}$ and $\theta \in S^1$ so that $(r, \theta) \in \mathbb{R}^2$ are polar coordinates, and $z \in \mathbb{R}$.

   (a) (10 points) Consider $\alpha = \cos(r)dz + r\sin(r)d\theta$.
   Show that the 3-form $\alpha d\alpha \in \Omega^3(\mathbb{R}^3)$ is no-where zero.

   (b) (5 points) Find the value $\alpha d\alpha(\partial_x, \partial_y, \partial_z)$, where $\partial_x, \partial_y, \partial_z \in T_0\mathbb{R}^3$ is the Cartesian axial basis of the tangent space of $0 \in \mathbb{R}^3$.

(c) (10 points) Let $\mathbb{R}^2_{x,y}$ have coordinates $(x, y) \in \mathbb{R}^2$. Consider $\mathbb{R}^5 = \mathbb{R}^3_{r,\theta,z} \times \mathbb{R}^2_{x,y}$ and

$$\lambda = xdy - ydx \in \Omega^1(\mathbb{R}^2).$$

Compute $\eta(d\eta)^2 \in \Omega^5(\mathbb{R}^5)$ where $\eta = \alpha + \lambda$.

2. (25 points) Consider $X = \mathbb{R}^3 \setminus \{0\}$ with Cartesian coordinates (x, y, z), and the 2-form

$$\omega = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \cdot (xdydz + ydzdx + zdxdy) \in \Omega^2(X).$$

(a) (10 points) Show that $d\omega$=0.

(b) (15 points) Prove that ω is not exact, i.e. $\nexists \eta \in \Omega^1(X)$ such that $d\eta = \omega$.

3. (25 points) Consider the 2-form $\omega = dxdy + dydz \in \Omega^2(\mathbb{R}^3)$ and the unit 2-sphere

$$S^2 = \{(\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), \cos(\varphi)) \in \mathbb{R}^3 : (\theta, \varphi) \in [0, 2\pi) \times [0, \pi]\} \subseteq \mathbb{R}^3$$

parametrized with spherical coordinates $(\theta, \varphi) \in [0, 2\pi) \times [0, \pi]$. Denote by $i : S^2 \longrightarrow \mathbb{R}^3$ the inclusion map.

(a) (10 points) Compute the restriction $i^*\omega \in \Omega^2(S^2)$.

(b) (15 points) Show that $\int_{S^2} \omega = 0$.

4. (25 points) Let $T^2 = S^1 \times S^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ with coordinates $(\theta_1, \theta_2) \in T^2$. Consider the smooth map

$$f : T^2 \longrightarrow T^2, \quad f(\theta_1, \theta_2) = (4\theta_1 + 5\theta_2, 2\theta_1 + 3\theta_2),$$

and the 2-form $\eta = d\theta_1 d\theta_2 \in \Omega^2(T^2)$.

(a) (10 points) Compute the integral $\displaystyle\int_{T^2} f^*\eta$.

(b) (15 points) Show that f is not homotopic to g, where

$$g : T^2 \longrightarrow T^2, \quad g(\theta_1, \theta_2) = (\theta_1 + \theta_2, \theta_1 + 23\theta_2).$$