MAT 239: PROBLEM SET 3

DUE TO FRIDAY OCT 25 2024

Abstract. This problem set corresponds to the third week of the course MAT-239 Fall 2024. It is due Friday Oct 25 at 10:00am submitted via Gradescope.

Task: Solve Problems 3 and 4 below and submit them through Gradescope by Friday Oct 25 at 10:00am. Be rigorous and precise in writing your solutions. Please try some of the other problems as well, though they will not be graded, and also work through the corresponding problems in the textbook.

Problem 1. For each of the following smooth functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, determine whether f is a Morse function or not:

(a) $\cos(x)$.
(b) $x^3 + \varepsilon \cdot x$, depending on the value of $\varepsilon \in \mathbb{R}$.
(c) $\operatorname{tr}\Big((e^{-x^2}, \cos(x)^2, -x^2)^t \cdot (1, 0, -1)\Big)$.
(d) $Ai(x) := \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + xt)dt$.

Problem 2. Two Morse functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are said to be *equivalent* is they can be transformed one into the other by smooth changes of coordinates (a diffeomorphism) in the domain and in the target. Find the number $K(n)$ of pairwise non-equivalent functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ having n non-degenerate critical points with pairwise different critical values, supposing that at infinity the function behaves like x for even n and like x^2 for odd n.

(*Answer: The numbers $K(n)$ satisfy $\sum_n K(n)\frac{t^n}{n!} = \sec t + \tan t$.*)

Problem 3. Solve the following parts:

(a) Describe a Morse function in the 2-sphere S^2 with $2n$ critical points, where $n \in \mathbb{N}$ is a positive natural number.
(b) Find a non-constant smooth function $f : S^2 \longrightarrow \mathbb{R}$ which is not a Morse function.
(c) Is it possible for S^2 to have a Morse function with an odd number of critical points?

Problem 4. Let M, N be smooth manifolds and $f : M \longrightarrow \mathbb{R}$ and $g : N \longrightarrow \mathbb{R}$ be Morse functions.

(1) Show that $f + g : M \times N \longrightarrow \mathbb{R}$ is a Morse function.
(2) Compute the number of critical points and indices of $f + g$ in terms of the number of critical points and their indices of f and g.
(3) Construct a Morse function on the n-torus $T^n = S^1 \times \overset{(n)}{\cdots} \times S^1$ with $\binom{n}{k}$ critical points of index k (for a total of 2^n critical points).
(4) Find a smooth function $f : T^2 \longrightarrow \mathbb{R}$ with 3 critical points.

Problem 5. Consider the special orthogonal group

$$SO(n) := \{A \in M_{n\times n}(\mathbb{R}) : A \cdot A^t = \mathrm{Id}, \quad \det(A) = 1\}.$$

Let $R \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix with *distinct* positive eigenvalues and consider the function

$$f_R : SO(n) \longrightarrow \mathbb{R}, \quad f_R(A) = -\operatorname{tr}(R \cdot A^t).$$

(i) Show that f_R is a Morse function.
(ii) Compute the number of critical points of f_R and their indices.

Problem 6. (Classification of 1-dimensional smooth manifolds) Show that any smooth, connected 1-dimensional manifold X is diffeomorphic either to the circle S^1 or to some interval $I \subseteq \mathbb{R}$ of the real numbers.

(It follows that there are only four distinct connected 1-manifolds, up to diffeomorphism.)

Hint: consider a Morse function $f : X \longrightarrow \mathbb{R}$ and let S be the set of critical points of f union with the boundary points of X. Show that the complement of S in X consists of a finite number of connected 1-manifolds such that f maps each of them diffeomorphically onto and open interval of $\mathbb{R}$.

Problem 7. Show that if X is a compact manifold and $f : X \longrightarrow \mathbb{R}$ is a smooth Morse function with only two critical points, then X is homeomorphic to a sphere.

Problem 8. (If comfortable with cotangent bundle T^*X) Show that a smooth function $f : X \longrightarrow \mathbb{R}$ on a smooth manifold X is Morse if and only if the graph of $df : X \longrightarrow T^*X$ is transverse to the zero section.