

MAT 22B: PROBLEM SET 1

DUE TO FRIDAY OCT 2 AT 11:00AM

ABSTRACT. This is the first problem set for the Differential Equations Course in the Fall Quarter 2026. It is due Friday Oct 2 at 11:00am via online submission.

Purpose: The goal of this assignment is to practice the basic techniques for solving first-order differential equations. In particular, we would like to become familiar with method of **integrating factors** and **separation of variables**.

Task: Solve Problems 1 through 7 below. The first 2 problems and Problem 7 will not be graded but I trust that you will work on them. Versions of either of them can appear in the exams. Problems 3 to 6 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems.¹ However, you should write the solutions on your own, using your own words and thought process. In particular, it is not allowed to have an AI write the solutions for you in any form. Failure to adhere to such instructions results in an automatic failing grade.

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use “Elementary Differential Equations and Boundary Value Problems” by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $t \in \mathbb{R}$ be a real number and $y(t)$ a differentiable real valued function. Find all the solutions to the following differential equations:

(a) $y'(t) = \frac{1}{1+t^2},$

(b) $y'(t) = 3t^2 - 4 + \frac{1}{t} - \frac{7}{t^2},$ for $t > 0,$

(c) $y'(t) = 5y(t),$

(d) $y''(t) = -y(t).$

¹If you do, please list any collaborators in the upper-left corner of the first page.

Solution:

(a) Integrating both sides,

$$y(t) = \int \frac{1}{1+t^2} dt = \arctan(t) + C.$$

Thus,

$$y(t) = \arctan(t) + C, \quad C \in \mathbb{R}.$$

(b) Integrating term by term,

$$y(t) = \int \left(3t^2 - 4 + \frac{1}{t} - \frac{7}{t^2} \right) dt.$$

Hence,

$$y(t) = t^3 - 4t + \ln t + \frac{7}{t} + C, \quad t > 0.$$

(c) The equation is separable:

$$y'(t) = 5y(t).$$

For $y \neq 0$,

$$\frac{1}{y} dy = 5 dt.$$

Integrating,

$$\ln |y| = 5t + C_1.$$

Exponentiating gives

$$y(t) = Ce^{5t}.$$

The solution $y(t) = 0$ is included by taking $C = 0$. Therefore,

$$y(t) = Ce^{5t}, \quad C \in \mathbb{R}.$$

(d) Rewrite the equation as

$$y''(t) + y(t) = 0.$$

Try a solution of the form

$$y(t) = e^{rt}.$$

Then

$$y''(t) = r^2 e^{rt}.$$

Substitution gives

$$r^2 e^{rt} + e^{rt} = 0,$$

so

$$r^2 + 1 = 0.$$

The roots are

$$r = \pm i.$$

Therefore, the general real-valued solution is

$$y(t) = C_1 \cos t + C_2 \sin t,$$

where $C_1, C_2 \in \mathbb{R}$.

Problem 2. Let $t \in \mathbb{R}$, and consider the function

$$y(t) = \arccos(t) + 4t^2.$$

- (a) Find a first order differential equation which has $y(t)$ as a solution,
- (b) Find a second order differential equation which has $y(t)$ as a solution.

Solution:

Since $\arccos(t)$ is differentiable for $-1 < t < 1$, we work on this interval.

- (a) Differentiate $y(t)$:

$$y'(t) = \frac{d}{dt} (\arccos(t) + 4t^2).$$

Using

$$\frac{d}{dt} \arccos(t) = -\frac{1}{\sqrt{1-t^2}},$$

we get

$$y'(t) = 8t - \frac{1}{\sqrt{1-t^2}}.$$

Thus, one first-order differential equation is

$$y'(t) = 8t - \frac{1}{\sqrt{1-t^2}}, \quad -1 < t < 1.$$

- (b) Differentiate again:

$$y'(t) = 8t - (1-t^2)^{-1/2}.$$

Using the chain rule,

$$\frac{d}{dt} [-(1-t^2)^{-1/2}] = -\frac{t}{(1-t^2)^{3/2}}.$$

Therefore,

$$y''(t) = 8 - \frac{t}{(1-t^2)^{3/2}}.$$

One second-order differential equation is

$$y''(t) = 8 - \frac{t}{(1-t^2)^{3/2}}, \quad -1 < t < 1.$$

Problem 3. (25 pts) Consider the following differential equation:

$$y'(t) - \frac{3y(t)}{t+1} = (1+t)^4, \quad t > 0.$$

- (a) Find all the solutions to the differential equation above.
- (b) Find all the solutions which satisfy $y(0) = 3$.
- (c) Are there any solutions $y(t)$ such that $y(0) = 3$ and $y(1) = 100$?
- (d) Are there any solutions $y(t)$ such that $y(0) = 3$ and $y(1) = 36$?

Solution:

- (a) This is a first-order linear equation

$$y'(t) - \frac{3}{t+1}y(t) = (t+1)^4.$$

The integrating factor is

$$\mu(t) = e^{\int -\frac{3}{t+1} dt} = e^{-3 \ln(t+1)} = (t+1)^{-3}.$$

Multiplying the equation by $(t+1)^{-3}$ gives

$$(t+1)^{-3}y'(t) - 3(t+1)^{-4}y(t) = t+1.$$

The left side can be written as

$$\frac{d}{dt} [(t+1)^{-3}y(t)] = t+1.$$

Integrating,

$$(t+1)^{-3}y(t) = \frac{(t+1)^2}{2} + C.$$

Multiplying by $(t+1)^3$,

$$y(t) = \frac{(t+1)^5}{2} + C(t+1)^3, \quad C \in \mathbb{R}.$$

- (b) Using

$$y(t) = \frac{(t+1)^5}{2} + C(t+1)^3$$

and the condition $y(0) = 3$,

$$3 = \frac{1}{2} + C.$$

Thus,

$$C = \frac{5}{2}.$$

The solution is

$$y(t) = \frac{(t+1)^5}{2} + \frac{5}{2}(t+1)^3.$$

(c) Using the solution from part (b),

$$y(1) = \frac{2^5}{2} + \frac{5}{2}(2^3).$$

Thus,

$$y(1) = 16 + 20 = 36.$$

Since

$$36 \neq 100,$$

there is no solution satisfying both

$$y(0) = 3 \quad \text{and} \quad y(1) = 100.$$

(d) From part (c),

$$y(1) = 36.$$

Therefore, there is a solution satisfying both conditions. It is

$$y(t) = \frac{(t+1)^5}{2} + \frac{5}{2}(t+1)^3.$$

Problem 4. (25 pts) Consider the following differential equation:

$$y'(t) + 2ty(t) = t, \quad t > 0.$$

- (a) Find all the solutions to the differential equation above.
- (b) Let $y_1(t)$ be the unique solution such that $y_1(2) = 0.5$, $y_2(t)$ the unique solution such that $y_2(0) = 1$, and $y_3(t)$ the unique solution such that $y_3(0) = 2$. Compute the long term behaviours

$$\lim_{t \rightarrow \infty} y_1(t), \quad \lim_{t \rightarrow \infty} y_2(t), \quad \lim_{t \rightarrow \infty} y_3(t)$$

for these three solutions.

- (c) Plot the graph of the functions $y_1(t)$, $y_2(t)$ and $y_3(t)$.
- (d) Is there any solution $y(t)$ which tends to ∞ in its long term behaviour ?

Solution:

- (a) The equation

$$y'(t) + 2ty(t) = t$$

is a first-order linear differential equation.

The integrating factor is

$$\mu(t) = e^{\int 2t dt} = e^{t^2}.$$

Multiplying the equation by e^{t^2} gives

$$e^{t^2} y'(t) + 2te^{t^2} y(t) = te^{t^2}.$$

Thus,

$$\frac{d}{dt} (e^{t^2} y(t)) = te^{t^2}.$$

Integrating,

$$e^{t^2} y(t) = \int te^{t^2} dt.$$

Let

$$u = t^2, \quad du = 2t dt.$$

Then

$$\int te^{t^2} dt = \frac{1}{2} \int e^u du = \frac{1}{2} e^{t^2} + C.$$

Hence,

$$e^{t^2} y(t) = \frac{1}{2} e^{t^2} + C,$$

and

$$y(t) = \frac{1}{2} + Ce^{-t^2}, \quad C \in \mathbb{R}.$$

(b) The general solution is

$$y(t) = \frac{1}{2} + Ce^{-t^2}.$$

For y_1 ,

$$y_1(2) = \frac{1}{2},$$

so

$$\frac{1}{2} = \frac{1}{2} + Ce^{-4}.$$

Thus,

$$C = 0$$

and

$$y_1(t) = \frac{1}{2}.$$

Therefore,

$$\lim_{t \rightarrow \infty} y_1(t) = \frac{1}{2}.$$

For y_2 ,

$$y_2(0) = 1.$$

Thus,

$$1 = \frac{1}{2} + C,$$

so

$$C = \frac{1}{2}.$$

Hence,

$$y_2(t) = \frac{1}{2} + \frac{1}{2}e^{-t^2}.$$

Since

$$\lim_{t \rightarrow \infty} e^{-t^2} = 0,$$

we have

$$\lim_{t \rightarrow \infty} y_2(t) = \frac{1}{2}.$$

For y_3 ,

$$y_3(0) = 2.$$

Thus,

$$2 = \frac{1}{2} + C,$$

which gives

$$C = \frac{3}{2}.$$

Hence,

$$y_3(t) = \frac{1}{2} + \frac{3}{2}e^{-t^2},$$

and

$$\lim_{t \rightarrow \infty} y_3(t) = \frac{1}{2}.$$

Thus,

$$\lim_{t \rightarrow \infty} y_1(t) = \lim_{t \rightarrow \infty} y_2(t) = \lim_{t \rightarrow \infty} y_3(t) = \frac{1}{2}.$$

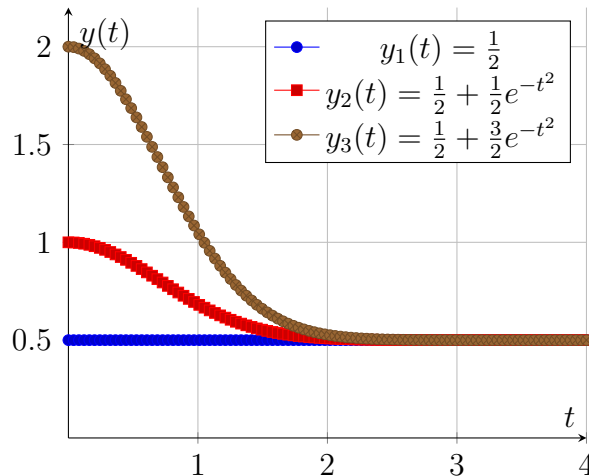
(c) The three solutions are

$$y_1(t) = \frac{1}{2},$$

$$y_2(t) = \frac{1}{2} + \frac{1}{2}e^{-t^2},$$

and

$$y_3(t) = \frac{1}{2} + \frac{3}{2}e^{-t^2}.$$



(d) Every solution has the form

$$y(t) = \frac{1}{2} + Ce^{-t^2}.$$

Since

$$\lim_{t \rightarrow \infty} e^{-t^2} = 0,$$

for every $C \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} y(t) = \frac{1}{2}.$$

Therefore, there is no solution that tends to ∞ as $t \rightarrow \infty$.

PROBLEM 5: 25 POINTS

Given:

$$y'(t) = y(t)^2 \frac{t-3}{t^3}, \quad t > 0.$$

(a) Find infinitely many solutions (12 points). First separate the variables for a nonzero solution: (4 points)

$$\frac{dy}{y^2} = (t^{-2} - 3t^{-3}) dt.$$

Integrate both sides: (4 points)

$$(0.1) \quad -\frac{1}{y} = -\frac{1}{t} + \frac{3}{2t^2} + K.$$

Note the term on the right is positive because $\int -3t^{-3} dt = \frac{3}{2}t^{-2}$, and the inclusion of the arbitrary constant K .

Next multiply by -1 and rename the arbitrary constant $C = -K$: (1 point)

$$\frac{1}{y} = C + \frac{1}{t} - \frac{3}{2t^2}.$$

Solve for y and exhibit an infinite family: (3 points)

$$y(t) = \frac{1}{C + 1/t - 3/(2t^2)} = \frac{2t^2}{2Ct^2 + 2t - 3}.$$

Each unique value of C produces a solution for which $t > 0$ and the denominator is nonzero. For example, $C = 0$ gives a distinct solution for $t > 0$ and $t \neq \frac{3}{2}$, so there are infinite solutions.

Verification: For a check, let $D = C + 1/t - 3/(2t^2)$. Then $D' = -1/t^2 + 3/t^3$, so

$$y' = -\frac{D'}{D^2} = y^2 \frac{t-3}{t^3}.$$

(b) Solve the initial value problem $y(1) = -1$ (10 points). Impose the initial condition and determine the constant: (5 points)

$$-1 = C + 1 - \frac{3}{2} = C - \frac{1}{2}, \quad C = -\frac{1}{2}.$$

Substitute the constant to obtain the particular solution: (5 points)

$$y(t) = \frac{2t^2}{-t^2 + 2t - 3} = -\frac{2t^2}{(t-1)^2 + 2}.$$

Verification: The denominator $(t-1)^2 + 2$ is always positive, so the maximal interval in the stated region is $(0, \infty)$. Substitution gives $y(1) = -2/2 = -1$, and the derivative check above verifies the equation.

(c) Is $y = 0$ a solution? (3 points). Note there is no restriction on the domain of y , thus 0 is a valid value for y . If $y = 0$, then $dy = 0$, which makes $y = 0$ a solution.

Answered yes with justification: 3 points

Answered yes without justification: 2 points

Answered no with justification: 1 point

Answered no without justification: 0.5 points

Blank: 0 points

PROBLEM 6: 25 POINTS

Given:

$$y'(t) = 2t(1 - y(t))^2, \quad t \in \mathbb{R}.$$

(a) Find all solutions (10 points). First identify the equilibrium: (1 point)

$$y(t) = 1, \quad y' = 0 = 2t(1 - 1)^2.$$

Next separate the variables for $y \neq 1$: (3 points)

$$\frac{dy}{(1 - y)^2} = 2t \, dt.$$

Integrate both sides, including the arbitrary constant: (3 points)

The derivative of $(1 - y)^{-1}$ with respect to y is positive:

$$\frac{1}{1 - y} = t^2 + C.$$

Solve for y : (3 points)

$$y(t) = 1 - \frac{1}{t^2 + C}, \quad \text{and } y(t) = 1.$$

The nonconstant formulas apply only where $t^2 + C \neq 0$.

The right side and its y derivative are continuous, so local uniqueness applies. A solution meeting $y = 1$ cannot then leave that equilibrium. Thus the list is complete.

Maximal intervals:

$C > 0$: all of $\mathbb{R}$.

$C = 0$: $(-\infty, 0)$ or $(0, \infty)$.

$C < 0$: with $a = \sqrt{-C}$, the intervals are $(-\infty, -a)$, $(-a, a)$, and (a, ∞) .

If “solutions” is interpreted as functions defined on all of $\mathbb{R}$, only the $C > 0$ family and $y = 1$ qualify. The other branches are legitimate local solutions.

(b) Can $y_1(2) = 1$ and $y_1(3) = 1$? (3 points). First identify a candidate that takes the value 1 at both times: (1 point)

$$y_1(t) = 1.$$

This notation means that $y_1(t)$ equals 1 for every t , not just at $t = 2$ and $t = 3$. A constant function is a natural candidate because the factor $(1 - y)^2$ vanishes when $y = 1$.

Next verify that this candidate solves the differential equation: (1 point)

$$y_1'(t) = \frac{d}{dt}(1) = 0, \quad 2t(1 - y_1(t))^2 = 2t(1 - 1)^2 = 0.$$

Both sides equal zero for every real t , so the constant function is a solution on all of $\mathbb{R}$. Merely taking the specified values would not be enough: the function must also satisfy the differential equation.

Finally check both prescribed values and answer the existence question: (1 point)

$$y_1(2) = 1, \quad y_1(3) = 1, \quad \boxed{\text{Yes, } y_1(t) = 1 \text{ satisfies both conditions.}}$$

The two conditions are compatible because the same solution satisfies both. There is no requirement that a solution change with time.

The nonconstant family from (a), $y(t) = 1 - 1/(t^2 + C)$, cannot equal 1 at a finite time where it is defined: that would require $1/(t^2 + C) = 0$, which is impossible for a finite real C . The constant solution must therefore be retained separately; dividing by $(1 - y)^2$ during separation excludes it. This is why a search only within the nonconstant formula would miss the solution needed here.

Answered yes with justification: 3 points

Answered yes without justification: 2 points

Answered no with justification: 1 point

Answered no without justification: 0.5 points

Blank: 0 points

(c) Qualitatively plot six different solutions (7 points). Choose the equilibrium and five members of the global family:

$$y = 1, \quad y_C(t) = 1 - \frac{1}{t^2 + C}, \quad C \in \left\{ \frac{1}{4}, \frac{1}{2}, 1, 2, 4 \right\}.$$

To sketch them:

Draw the equilibrium curve $y = 1$: (0.5 point)

Draw five distinct nonconstant solution curves (0.5 each): (2.5 points) Identify them using their minima at $t = 0$:

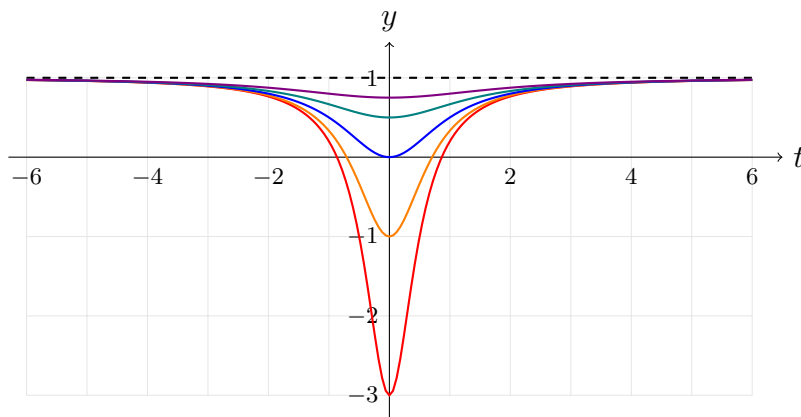
$$y_C(0) = 1 - \frac{1}{C} = -3, -1, 0, \frac{1}{2}, \frac{3}{4},$$

respectively.

Each nonconstant curve is even, so reflect its positive-time half across the vertical axis.

Show the correct direction of change: (2 points) Since $y' = 2t/(t^2 + C)^2$, the curves decrease for $t < 0$, have horizontal tangent at $t = 0$, and increase for $t > 0$.

Show the correct limiting behavior: (2 points) Draw smooth valleys approaching $y = 1$ from below at both ends. None crosses the equilibrium.



Red: $C = 1/4$; orange: $C = 1/2$; blue: $C = 1$; teal: $C = 2$; violet: $C = 4$; dashed: $y = 1$.

(d) Long-term behavior (5 points). The equilibrium stays at 1: (1 point)

$$y(t) = 1 \implies \lim_{t \rightarrow \infty} y(t) = 1.$$

Determine the limit of every branch extending to $+\infty$: (3 points)

Every branch extending to arbitrarily large positive times satisfies

$$\lim_{t \rightarrow \infty} \left(1 - \frac{1}{t^2 + C} \right) = 1,$$

approaching from below. Branches extending to $-\infty$ likewise approach 1 from below.

Distinguish branches ending in finite time, which have no limit at $+\infty$: (1 point)

Their endpoint behavior is:

$C = 0$: $y \rightarrow -\infty$ as $t \rightarrow 0^-$ on the negative branch, and as $t \rightarrow 0^+$ on the positive branch.

$C < 0$, $a = \sqrt{-C}$: the left branch tends to $-\infty$ as $t \rightarrow -a^-$. The middle branch tends to $+\infty$ as $t \rightarrow -a^+$ and as $t \rightarrow a^-$. The right branch tends to $-\infty$ as $t \rightarrow a^+$, and approaches 1 as $t \rightarrow \infty$.

Thus all *global* solutions approach 1, but one cannot assign an infinite-time limit to a branch that blows up earlier.

PROBLEM 7: (25 POINTS, NOT GRADED)

Given:

$$y' = y(1 - y), \quad t \in \mathbb{R}^+.$$

Since part (b) specifies $y(0)$, interpret the initial value through the continuous extension to $t = 0$.

(a) Find all solutions (9 points). First identify the zero equilibrium: (1.5 points)

$$y(t) = 0, \quad y' = 0 = 0(1 - 0).$$

Next identify the other equilibrium: (1.5 points)

$$y(t) = 1, \quad y' = 0 = 1(1 - 1).$$

Separate for $y \neq 0, 1$: (1.5 points)

$$\frac{dy}{y(1 - y)} = dt.$$

Use correct partial fractions: (1.5 points)

$$\frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y}.$$

Integrate with a constant: (1.5 points)

$$\ln |y| - \ln |1 - y| = t + K, \quad \ln \left| \frac{y}{1 - y} \right| = t + K.$$

Solve for the logistic family: (1.5 points) Exponentiate with a *signed*, nonzero constant A and solve:

$$\frac{y}{1 - y} = Ae^t, \quad y = \frac{Ae^t}{1 + Ae^t} = \frac{1}{1 + Ce^{-t}}, \quad C = 1/A.$$

Allowing $C = 0$ includes $y = 1$. Retain $y = 0$ separately. Each formula is used on a connected interval where its denominator is nonzero.

Local uniqueness prevents solutions from meeting an equilibrium and leaving it, so there are no additional solutions obtained by splicing branches.

For $y(0) = y_0$, the convenient initial-value form is

$$y(t) = \frac{y_0 e^t}{1 - y_0 + y_0 e^t}.$$

It includes both equilibria and applies on the interval containing zero where the denominator does not vanish.

(b) Long-term behavior in terms of y_0 (7 points). The sign of $y(1 - y)$ explains the motion:

$y_0 = 0$: the solution remains zero, with limit zero. (1 point)

$0 < y_0 < 1$: $y' > 0$, so the solution increases toward 1 from below. (2 points)

$y_0 = 1$: the solution remains 1. (1 point)

$y_0 > 1$: $y' < 0$, so the solution decreases toward 1 from above. (2 points)

In particular, every positive initial value gives limit 1. This also follows by dividing numerator and denominator by e^t :

$$y(t) = \frac{y_0}{(1 - y_0)e^{-t} + y_0} \longrightarrow 1 \quad (y_0 > 0).$$

For $y_0 < 0$, identify decreasing behavior and finite-time blow-up, so no infinite-time limit exists: (1 point)

If $y_0 < 0$, the derivative is negative and the denominator vanishes at

$$t_* = \ln \left(\frac{y_0 - 1}{y_0} \right) = \ln \left(1 - \frac{1}{y_0} \right) > 0.$$

As $t \rightarrow t_*^-$, the denominator tends to 0^+ and the numerator is negative, so $y \rightarrow -\infty$. This initial-value solution has no $t \rightarrow \infty$ limit. Negative values are mathematically allowed but do not represent population sizes.

(c) Compare population models (5 points). Contrast unbounded exponential growth with logistic saturation: (2 points)

The exponential equation $y' = y$ has solution $y = y_0 e^t$, so every positive initial population grows without bound.

Explain density dependence and carrying capacity: (2 points)

Logistic growth has per-capita rate $1 - y$ and normalized carrying capacity 1: populations below 1 grow, while those above 1 decline.

Justify the accuracy comparison: (1 point)

For a population with limited resources, logistic growth is usually more realistic over longer times. Exponential growth may approximate early growth or abundant-resource conditions, since $1 - y \approx 1$ when y is small. Neither model is universally more accurate; accuracy depends on the system and data. A fixed carrying capacity is itself a simplifying assumption.

(d) Effect of $\lambda > 0$ (4 points). Separate and integrate the parameterized equation: (1 point)

$$\ln \left| \frac{y}{1 - y} \right| = \lambda t + K,$$

Solve for the initial-value solution: (1 point)

$$y_\lambda(t) = \frac{y_0 e^{\lambda t}}{1 - y_0 + y_0 e^{\lambda t}}.$$

Explain the time-scale change: (1 point)

For the same initial value, $y_\lambda(t) = y_{\lambda=1}(\lambda t)$: larger positive λ compresses the time scale, so the same change happens sooner.

Explain unchanged equilibria and positive-initial-value limits: (1 point)

Both equilibria and the limiting value 1 for positive initial values are unchanged. For $y_0 < 0$, the blow-up time is

$$t_* = \frac{1}{\lambda} \ln \left(1 - \frac{1}{y_0} \right).$$

Parameter clarification: The displayed equation specifies $\lambda \in \mathbb{R}^+$, but the final sentence asks about $\lambda \in \mathbb{R}$. If zero and negative parameters are intended:

$\lambda = 0$: every solution is constant at its initial value.

$\lambda < 0$: the trajectories run through logistic behavior in reverse. For $y_0 < 1$, the forward limit is zero (from below for $y_0 < 0$, from above for $0 < y_0 < 1$); $y_0 = 0$ remains zero and $y_0 = 1$ remains 1. For $y_0 > 1$, the solution increases to $+\infty$ at $t_* = \ln(1 - 1/y_0)/\lambda > 0$. Thus 0 is stable and 1 is unstable.