MAT 22B: PROBLEM SET 2

DUE TO FRIDAY OCT 9 AT 11:00AM

Abstract. This is the second problem set for the Differential Equations Course in the Fall Quarter 2026. It is due Friday Oct 9 at 11:00am via Gradescope submission.

Purpose: The goal of this assignment is to practice the basic techniques for qualitatively and numerically solving first-order differential equations. In particular, we would like to become familiar with method of **direction fields**, the study of **autonomous differential equations** and **Euler's Method**.

Task: Solve Problems 1 through 7 below. The first 2 problems will not be graded but I trust that you will work on them. Problems 3 to 7 will be graded.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade: Each graded Problem is worth 20 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Elementary Differential Equations and Boundary Value Problems" by W.E. Boyce, R.C. DiPrima and D.B. Meade (11th Edition). Please contact me if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. For each of the following differential equations, qualitatively plot their direction field in the (t, y)-plane:

(a) $y'(t) = y(t)(4 - y(t))$,

(b) $y'(t) = y(t)^2$,

(c) $y'(t) = \sin^2(t)$,

(d) $y'(t) = \dfrac{t^2}{1 - y(t)^2}$.

Problem Problem 1 Solution: Direction fields

A direction field places a short line segment at each point (t, y). Its slope is the value of y' there. Read every segment from left to right: a positive slope rises, a negative slope falls, and a zero slope is horizontal. The segments show the direction a solution curve follows.

(a) $y' = y(4 - y)$

Find the rows with zero slope by setting the right side equal to zero:

$$y(4 - y) = 0 \quad \Longrightarrow \quad y = 0 \text{ or } 4 - y = 0 \quad \Longrightarrow \quad y = 0 \text{ or } y = 4.$$

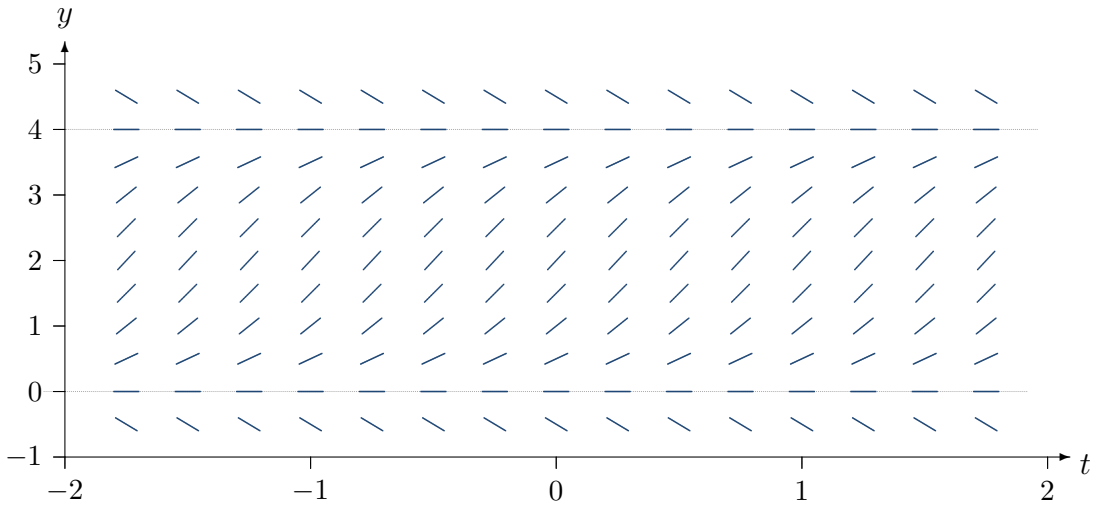
Check the sign of each factor on the three remaining ranges:

Height y	Sign of y	Sign of $4 - y$	Slope $y(4 - y)$
$y < 0$	$-$	$+$	$-$ (falls)
$0 < y < 4$	$+$	$+$	$+$ (rises)
$y > 4$	$+$	$-$	$-$ (falls)

For the steepness, some sample slopes are

$$\begin{aligned} y = -1: &\quad (-1)(4 - (-1)) = (-1)(5) = -5, \\ y = 1: &\quad 1(4 - 1) = 1(3) = 3, \\ y = 2: &\quad 2(4 - 2) = 2(2) = 4, \\ y = 5: &\quad 5(4 - 5) = 5(-1) = -5. \end{aligned}$$

The formula does not contain t, so copy the same pattern across every horizontal row. Draw horizontal segments at $y = 0$ and $y = 4$.


y
5
4
3
2
1
0
−1
−2
−1
0
1
2
t

(b) $y' = y^2$

Squaring a real number never gives a negative number:

$$y^2 > 0 \quad \text{when } y \neq 0, \qquad y^2 = 0 \quad \text{when } y = 0.$$

Therefore every segment rises except those on $y = 0$, which are horizontal. Negative values of y still give positive slopes; for example,

$$(-2)^2 = (-2)(-2) = 4.$$

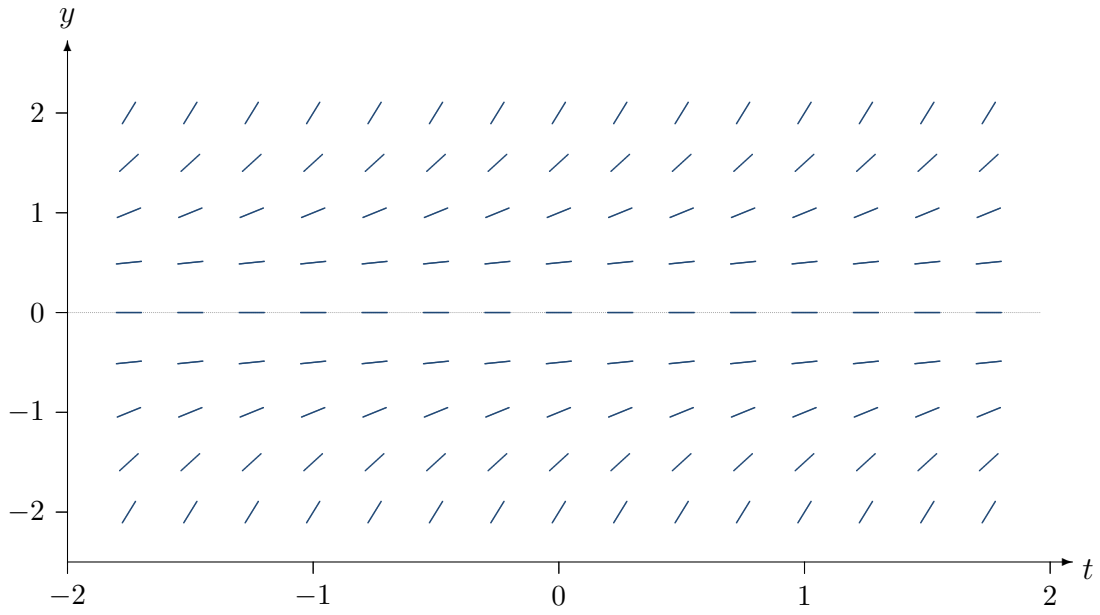
Compute sample slopes to decide how steep to draw the segments:

y	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y^2	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4

In particular,

$$\left(-\frac{1}{2}\right)^2 = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{4}, \qquad \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

The slopes at y and $-y$ are equal. Segments get steeper as the distance from the row $y = 0$ increases. The formula does not contain t, so the slopes repeat across each horizontal row.



The row $y = 0$ is also a constant solution: if $y(t) = 0$ for every t, then $y'(t) = 0$ and $y(t)^2 = 0$.

(c) $y' = \sin^2(t)$

Here $\sin^2(t)$ means $(\sin t)^2$. Since $-1 \le \sin t \le 1$,

$$0 \le (\sin t)^2 \le 1.$$

Thus, the segments are horizontal or rising and their slopes never exceed 1.

The zero slopes occur when

$$\sin^2(t) = 0 \quad \Longrightarrow \quad \sin t = 0 \quad \Longrightarrow \quad t = k\pi, \qquad k \in \mathbb{Z}.$$

Here $k \in \mathbb{Z}$ means k can be any integer, including zero and negative integers. The largest slope is 1, reached when

$$\sin t = 1 \text{ or } -1 \quad \Longrightarrow \quad t = \frac{\pi}{2} + k\pi.$$

Between these columns, use the sample values

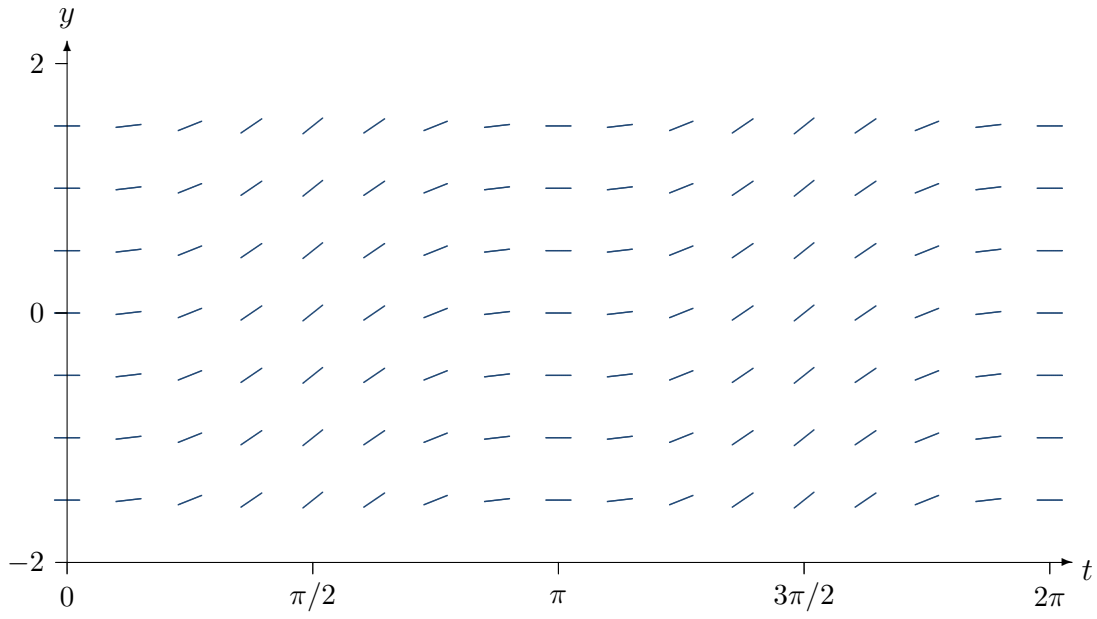
t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\sin t$	0	$\sqrt{2}/2$	1	$\sqrt{2}/2$	0
$\sin^2 t$	0	$1/2$	1	$1/2$	0

The fraction in the table comes from

$$\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{(\sqrt{2})^2}{2^2} = \frac{2}{4} = \frac{1}{2}.$$

The formula does not contain y, so copy each slope up and down its vertical column. The pattern repeats every π units because

$$\sin^2(t + \pi) = (-\sin t)^2 = \sin^2 t.$$



The horizontal segments at $t = k\pi$ do not give a constant solution: they occupy individual columns, and the slopes are positive at other times.

(d) $y' = \dfrac{t^2}{1-y^2}$

First find where the formula is undefined:

$$1 - y^2 = 0 \quad \Longrightarrow \quad y^2 = 1 \quad \Longrightarrow \quad y = -1 \text{ or } y = 1.$$

Draw no slope segments on either of these rows. This includes $(0, -1)$ and $(0, 1)$: the expression $0/0$ is undefined.

At $t = 0$ and any allowed height,

$$\frac{0^2}{1-y^2} = \frac{0}{1-y^2} = 0,$$

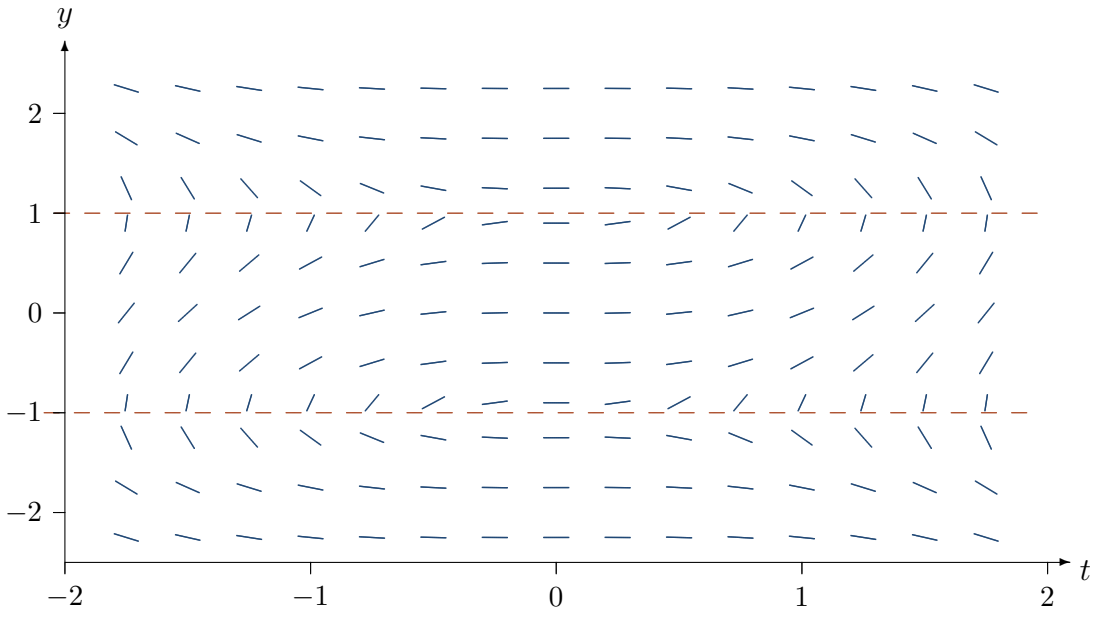
so the segments in that column are horizontal. For $t \neq 0$, the numerator t^2 is positive, so the denominator determines the sign:

Height y	Denominator $1 - y^2$	Slope for $t \neq 0$
$-1 < y < 1$	positive	positive (rises)
$y < -1$ or $y > 1$	negative	negative (falls)

For example, at $t = 1$,

$$\begin{aligned} y = 0: &\quad \frac{1^2}{1-0^2} = \frac{1}{1} = 1, \\ y = \tfrac{1}{2}: &\quad \frac{1}{1-(1/2)^2} = \frac{1}{1-1/4} = \frac{1}{3/4} = \frac{4}{3}, \\ y = 2: &\quad \frac{1}{1-2^2} = \frac{1}{1-4} = \frac{1}{-3} = -\frac{1}{3}. \end{aligned}$$

At a fixed $t \neq 0$, the segments become very steep near $y = \pm 1$, because the denominator approaches zero. At a fixed allowed y, replacing t by $2t$ multiplies the slope by 4, since $(2t)^2 = 4t^2$.



The dashed rows mark $y = \pm 1$, where the equation is undefined. The pattern is the same at t and $-t$, and at y and $-y$, because both variables are squared.

Problem 2. Consider the following differential equation:

$$y'(t) = y(t)^2 \cos(y(t)).$$

(a) Draw the direction field associated to this differential equation in the (t, y)-plane. (It should contain enough slopes such that the pattern is apparent.)

(b) Find the long-term behavior of *all* solutions $y(t)$ to this differential equation above in terms of their initial value $y(0)$.

(c) (Bonus) Give an example of a real-valued non-zero differentiable function $f(y)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, such that the differential equation $y'(t) = f(y(t))$ has at least a (continuous) interval worth of constant solutions, i.e. $\exists a, b \in \mathbb{R}$ such that the constant functions $y(t) \equiv c$ solve $y'(t) = f(y(t))$ for all $c \in (a, b)$.

Problem Problem 2 Solution: $y' = y^2 \cos y$

(a) Draw the direction field

The right side depends only on y, so slopes repeat along horizontal rows. Find the zero-slope rows:

$$y^2 \cos y = 0 \quad \Longrightarrow \quad y^2 = 0 \text{ or } \cos y = 0.$$

Solving these two equations gives

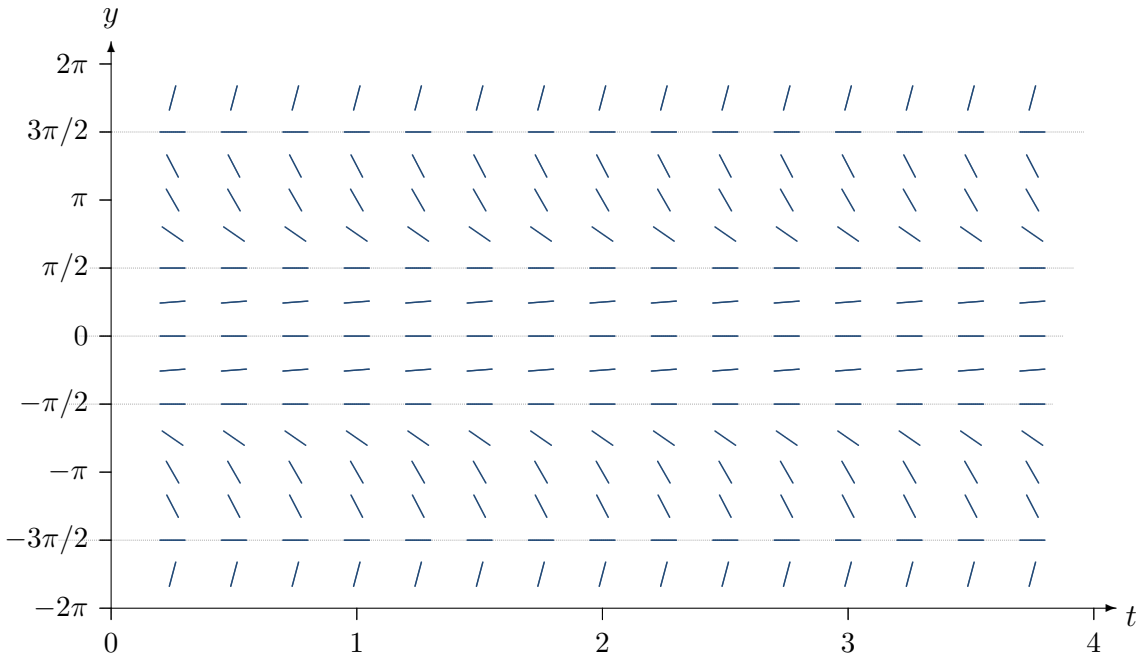
$$\boxed{y = 0 \quad \text{or} \quad y = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}.}$$

Each of these heights gives a constant solution. At any other height, $y^2 > 0$, so the slope has the same sign as $\cos y$. Near zero, for example,

Height interval	Sign of $\cos y$	Direction as t increases
$(-3\pi/2, -\pi/2)$	$-$	decreases
$(-\pi/2, 0)$	$+$	increases
$(0, \pi/2)$	$+$	increases
$(\pi/2, 3\pi/2)$	$-$	decreases

The cosine signs continue to alternate outside these intervals. The factor y^2 changes the steepness, so the full slope pattern is not periodic in y. For example,

$$\begin{aligned} y = \pi: &\quad \pi^2 \cos \pi = \pi^2(-1) = -\pi^2, \\ y = -\pi: &\quad (-\pi)^2 \cos(-\pi) = \pi^2(-1) = -\pi^2, \\ y = 2\pi: &\quad (2\pi)^2 \cos(2\pi) = 4\pi^2(1) = 4\pi^2. \end{aligned}$$


y
2π
3π/2
π
π/2
0
−π/2
−π
−3π/2
−2π
0
1
2
3
4
t

(b) Long-term behavior for every initial value

Write $y_0 = y(0)$. To track a solution, locate y_0 between the zero-slope rows. A positive slope moves y upward; a negative slope moves it downward.

The function $y^2 \cos y$ is smooth, so a solution cannot cross a constant solution. Between consecutive zero-slope rows, a moving solution stays between those rows and keeps moving in one direction. It must approach the row toward which it moves. It cannot stop at a height with nonzero slope, because there it would still be moving.

Near zero the directions are

$$-\frac{\pi}{2} \quad \longrightarrow \quad 0 \quad \longrightarrow \quad \frac{\pi}{2}.$$

These arrows show movement along the y-axis as time increases. Thus

$$\begin{aligned} -\frac{\pi}{2} < y_0 < 0 \quad &\Longrightarrow \quad y(t) \text{ increases toward } 0, \\ y_0 = 0 \quad &\Longrightarrow \quad y(t) = 0 \text{ for every } t, \\ 0 < y_0 < \frac{\pi}{2} \quad &\Longrightarrow \quad y(t) \text{ increases toward } \frac{\pi}{2}. \end{aligned}$$

To list all the other intervals without an infinite table, name the cosine zeros

$$a_k = \frac{\pi}{2} + k\pi.$$

For example, $a_{-2} = -3\pi/2$, $a_{-1} = -\pi/2$, $a_0 = \pi/2$, and $a_1 = 3\pi/2$. To find the cosine sign between a_k and a_{k+1}, test their midpoint:

$$\begin{aligned} \frac{a_k + a_{k+1}}{2} &= \frac{(\pi/2 + k\pi) + (\pi/2 + (k+1)\pi)}{2} \\ &= \frac{2\pi + 2k\pi}{2} = (k+1)\pi, \\ \cos((k+1)\pi) &= (-1)^{k+1}. \end{aligned}$$

For even k, this sign is negative, so the solution decreases toward a_k. For odd k, it is positive, so the solution increases toward a_{k+1}. The interval with $k = -1$ contains the extra zero at $y = 0$, so it is split into the two cases above.

Initial value	Limit as $t \to \infty$
$y_0 = 0$	0 (constant solution)
$y_0 = a_k$, any integer k	a_k (constant solution)
$-\pi/2 < y_0 < 0$	0
$0 < y_0 < \pi/2$	$\pi/2$
$a_k < y_0 < a_{k+1}$, k even	a_k
$a_k < y_0 < a_{k+1}$, k odd and $k \neq -1$	a_{k+1}

Here "even" includes $\ldots, -4, -2, 0, 2, 4, \ldots$ and "odd" includes $\ldots, -3, -1, 1, 3, \ldots$. Every starting value appears in exactly one case. Every solution stays bounded between constant solutions and exists for all future time.

(c) Bonus: An interval of constant solutions

For $y(t) = c$ to be a constant solution, we need

$$y'(t) = 0 = f(c).$$

Thus we want a differentiable function that is zero on an interval, but is not zero everywhere. Choose

$$\boxed{f(y) = \begin{cases} 0, & y \leq 0, \\ y^2, & y > 0. \end{cases}}$$

This function is not the zero function, because $f(1) = 1^2 = 1$.

Away from $y = 0$, each piece is differentiable. At the joining point, both pieces approach zero:

$$\lim_{y \to 0^-} f(y) = 0, \qquad \lim_{y \to 0^+} f(y) = \lim_{y \to 0^+} y^2 = 0 = f(0).$$

To check the derivative at zero, use its definition:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}.$$

From the left, $h < 0$, so $f(h) = 0$ and

$$\frac{f(h)}{h} = \frac{0}{h} = 0.$$

From the right, $h > 0$, so $f(h) = h^2$ and

$$\frac{f(h)}{h} = \frac{h^2}{h} = h \longrightarrow 0.$$

Both sides give the same limit, so $f'(0) = 0$. The function is differentiable on all of $\mathbb{R}$.

Finally, choose the interval $(a, b) = (-1, 0)$. For every $c \in (-1, 0)$,

$$y(t) = c \quad \Longrightarrow \quad y'(t) = 0, \qquad f(y(t)) = f(c) = 0.$$

Therefore every one of these constant functions solves the equation.

Here "non-zero function" means a function that is not zero everywhere. If it meant $f(y) \neq 0$ at every point, no constant solution would be possible.

Problem 3. (20 pts) Consider the following non-autonomous differential equation:

$$y'(t) = 2e^{t^2} + \frac{1}{5}\cos(y(t)), \quad t \geq 0.$$

(a) Plot qualitatively the direction field associated to this differential equation in the (t, y)-plane. The drawn direction field should contain drawn slopes such that the pattern is apparent, else use words to describe it.

(b) Show that the differential equation above does not admit any *constant* solution, i.e. a solution $y(t)$ is never a constant function.

(c) Describe the long-term behavior of *all* solutions to the differential equation.

Problem Problem 3 Solution: $y' = 2e^{t^2} + \frac{1}{5}\cos y$, $t \geq 0$ **(20 points)**

(a) Draw and describe the direction field (8 points)

Start with the range of the cosine (1 point):

$$-1 \leq \cos y \leq 1.$$

Multiplying each part by the positive number $1/5$ preserves both inequalities (1 point):

$$-\frac{1}{5} \leq \frac{1}{5}\cos y \leq \frac{1}{5}.$$

Add $2e^{t^2}$ throughout (1 point):

$$2e^{t^2} - \frac{1}{5} \leq y' \leq 2e^{t^2} + \frac{1}{5}.$$

Since $t^2 \geq 0$, we have $e^{t^2} \geq e^0 = 1$. Use this to obtain a positive lower bound (1 point):

$$y' \geq 2e^{t^2} - \frac{1}{5} \geq 2(1) - \frac{1}{5} = \frac{10}{5} - \frac{1}{5} = \frac{9}{5} > 0.$$

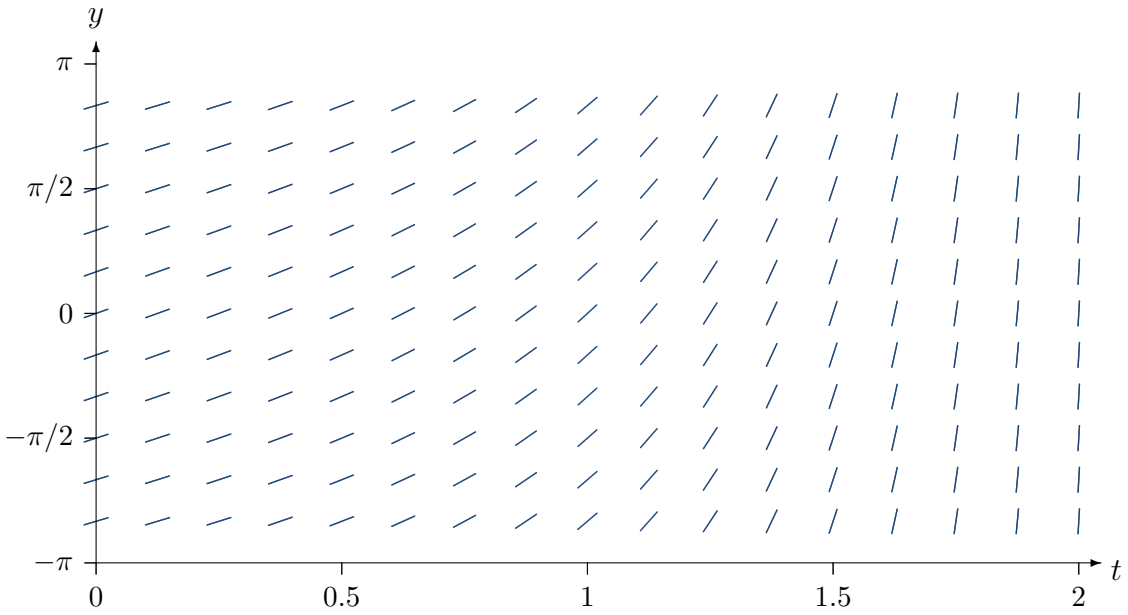
Every segment rises. In any fixed column, changing y changes the slope by at most $2/5$, the difference between $+1/5$ and $-1/5$.

At $t = 0$, the steepest and least steep slopes are (1 point)

$$\cos y = 1: \quad 2e^0 + \frac{1}{5} = 2 + \frac{1}{5} = \frac{11}{5},$$
$$\cos y = -1: \quad 2e^0 - \frac{1}{5} = 2 - \frac{1}{5} = \frac{9}{5}.$$

The steepest rows in a column are $y = 2k\pi$; the least steep are $y = (2k+1)\pi$. As t increases, e^{t^2} increases, so the segments become much steeper to the right. The pattern repeats vertically every 2π (1 point).

Draw the direction field using these slopes (2 points):


y
π
π/2
0
−π/2
−π
0
0.5
1
1.5
2
t

(b) Show that no constant solution exists (4 points)

Suppose a constant solution did exist, and write it as $y(t) = c$. A constant has derivative zero (1 point). Substitute it into the equation (1 point):

$$0 = 2e^{t^2} + \frac{1}{5}\cos c.$$

But part (a) shows that, for every $t \geq 0$ and every real c (1 point),

$$2e^{t^2} + \frac{1}{5}\cos c \geq \frac{9}{5} > 0.$$

The same number cannot be both zero and positive. Therefore (1 point),

$$\boxed{\text{There are no constant solutions.}}$$

(c) Describe the long-term behavior (8 points)

Let $y(0) = y_0$. From part (a), $y'(t) \geq 9/5$ for every $t \geq 0$ (1 point). Integrate from 0 to t (1 point):

$$\int_0^t y'(s)\,ds \geq \int_0^t \frac{9}{5}\,ds.$$

The left integral is the change in y; the right integral is a constant times the interval length (1 point). Simplify and add y_0 to both sides (1 point):

$$\begin{aligned} y(t) - y(0) &\geq \frac{9}{5}[s]_0^t, \\ y(t) - y_0 &\geq \frac{9}{5}(t - 0), \\ y(t) - y_0 &\geq \frac{9}{5}t, \\ y(t) &\geq y_0 + \frac{9}{5}t. \end{aligned}$$

The lower bound $y_0 + (9/5)t$ grows without limit as $t \to \infty$, regardless of the fixed starting value y_0 (2 points).

To check that a solution cannot become infinite at a finite time, fix any finite $T > 0$. For $0 \leq t \leq T$,

$$0 < y'(t) \leq 2e^{t^2} + \frac{1}{5} \leq 2e^{T^2} + \frac{1}{5}.$$

Integrating this upper bound gives

$$\begin{aligned} y(t) - y_0 &\leq \int_0^t \left(2e^{T^2} + \frac{1}{5}\right) ds \\ &= \left(2e^{T^2} + \frac{1}{5}\right) t \leq \left(2e^{T^2} + \frac{1}{5}\right) T. \end{aligned}$$

This is finite. The equation is smooth everywhere, and the solution stays finite on every finite time interval, so it continues for all $t \geq 0$. Since every solution stays above a lower bound that grows without limit, we conclude (2 points):

$$\boxed{\lim_{t\to\infty} y(t) = +\infty \quad \text{for every real initial value } y_0.}$$

Problem 4. (20 pts) Consider the following autonomous differential equation:

$$y'(t) = y(t)^4 - 6y(t)^3 + 11y(t)^2 - 6y(t).$$

(a) Plot qualitatively the direction field associated to this differential equation in the (t, y)-plane.

(b) Solve the initial value problem given by the above differential equation and the initial condition $y(0) = 3$.

(c) Find all the constant solutions to the differential equation.

(d) Describe whether each constant solution is stable, unstable or semistable.

Problem Problem 4 Solution: $y' = y^4 - 6y^3 + 11y^2 - 6y$ **(20 points)**

(a) Factor the expression and draw the direction field (6 points)

Factor out the common factor y:

$$y^4 - 6y^3 + 11y^2 - 6y = y\big(y^3 - 6y^2 + 11y - 6\big).$$

To factor the cubic, split $-6y^2$ into $-y^2 - 5y^2$ and split $11y$ into $5y + 6y$. Then group terms:

$$\begin{aligned} y^3 - 6y^2 + 11y - 6 &= y^3 - y^2 - 5y^2 + 5y + 6y - 6 \\ &= y^2(y-1) - 5y(y-1) + 6(y-1) \\ &= (y-1)(y^2 - 5y + 6). \end{aligned}$$

For the remaining quadratic, split $-5y$ into $-2y - 3y$:

$$\begin{aligned} y^2 - 5y + 6 &= y^2 - 2y - 3y + 6 \\ &= y(y-2) - 3(y-2) \\ &= (y-3)(y-2). \end{aligned}$$

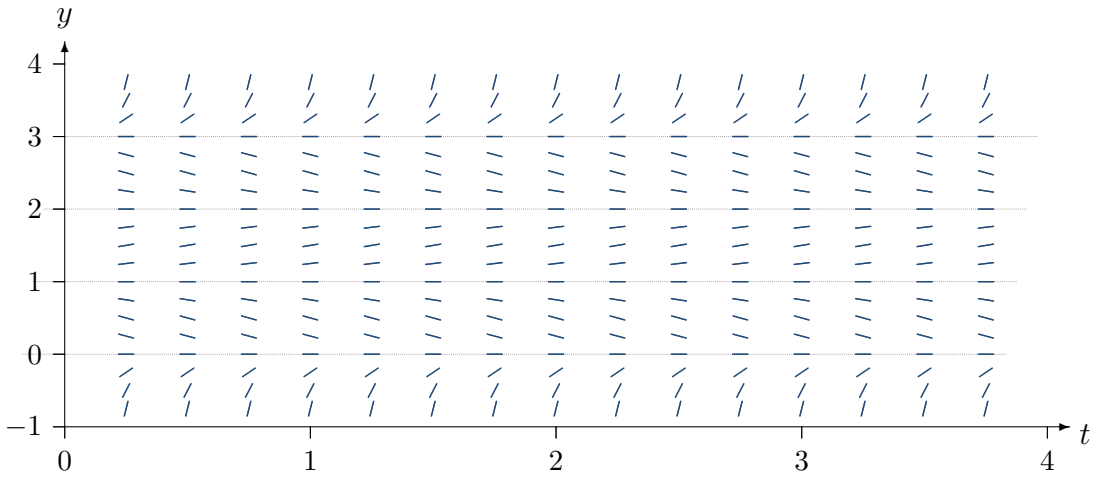
Putting the factors together and changing their order gives (1 point)

$$\boxed{y' = y(y-1)(y-2)(y-3).}$$

The slope is zero at $y = 0, 1, 2, 3$ (1 point). On each intervening interval, check the signs (2 points):

Height	y	$y-1$	$y-2$	$y-3$	Product
$y < 0$	$-$	$-$	$-$	$-$	$+$
$0 < y < 1$	$+$	$-$	$-$	$-$	$-$
$1 < y < 2$	$+$	$+$	$-$	$-$	$+$
$2 < y < 3$	$+$	$+$	$+$	$-$	$-$
$y > 3$	$+$	$+$	$+$	$+$	$+$

An even number of negative factors gives a positive product; an odd number gives a negative product. Draw rising segments for the $+$ rows, falling segments for the $-$ rows, and horizontal segments at $y = 0, 1, 2, 3$. Slopes repeat horizontally because the equation does not contain t (2 points).


y
4
3
2
1
0
−1
0
1
2
3
4
t

(b) Solve the initial value problem $y(0) = 3$ (4 points)

The initial height is one of the zero-slope rows. Identify the constant solution (2 points):

$$y(t) = 3.$$

Check its derivative and the right side of the equation (1 point):

$$y'(t) = \frac{d}{dt}(3) = 0, \qquad y(y-1)(y-2)(y-3) = 3(3-1)(3-2)(3-3).$$

Simplify the product:

$$3(3-1)(3-2)(3-3) = 3 \cdot 2 \cdot 1 \cdot 0 = 0.$$

Both sides are zero, and $y(0) = 3$, so the initial condition is satisfied (1 point). The right side is a polynomial and is smooth; this guarantees that two different solution curves cannot pass through the same initial point. Thus the solution is

$$\boxed{y(t) = 3 \quad \text{for all real } t.}$$

Separating variables by dividing by $y - 3$ would discard this solution, so checking constant solutions first matters here.

(c) Find all constant solutions (4 points)

For a constant function $y(t) = c$, the derivative is zero. Therefore

$$0 = c(c-1)(c-2)(c-3).$$

A product is zero exactly when at least one factor is zero:

$$c = 0, \quad c - 1 = 0, \quad c - 2 = 0, \quad \text{or} \quad c - 3 = 0.$$

Solving the four equations gives the constant solutions:

$$\boxed{\begin{array}{llll} y(t) = 0 & (1 \text{ point}), & y(t) = 1 & (1 \text{ point}), \\ y(t) = 2 & (1 \text{ point}), & y(t) = 3 & (1 \text{ point}). \end{array}}$$

(d) Classify the constant solutions (6 points)

A constant solution is *stable* if nearby solutions on both sides move toward it, *unstable* if they move away on both sides, and *semistable* if they move toward it on one side and away on the other. For these isolated constant solutions, the sign table gives the needed directions.

$$\longrightarrow \quad 0 \quad \longleftarrow \quad 1 \quad \longrightarrow \quad 2 \quad \longleftarrow \quad 3 \quad \longrightarrow$$

These arrows show movement in y as time increases, not movement in t. Classify each height using the movement on its two sides:

Constant	Just below it	Just above it	Classification
0	increases toward 0	decreases toward 0	stable (1.5 points)
1	decreases away from 1	increases away from 1	unstable (1.5 points)
2	increases toward 2	decreases toward 2	stable (1.5 points)
3	decreases away from 3	increases away from 3	unstable (1.5 points)

None is semistable. Although $y = 3$ is unstable, a solution starting *exactly* at 3 stays there; instability describes what happens after a small change in its starting value.

Problem 5. (20 pts) Let $f(y)$ be the real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ depicted in Figure 1, and consider the autonomous differential equation $y'(t) = f(y(t))$.

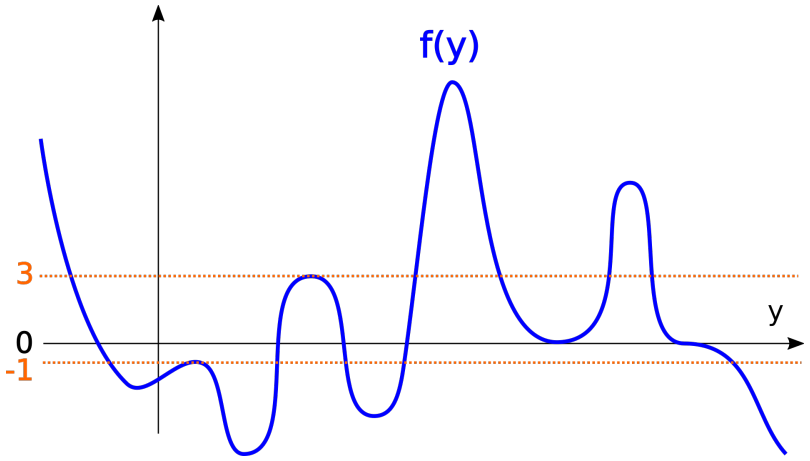


FIGURE 1. The function $f(y)$ for Problem 4.

(a) How many constant solutions does the above differential equation have ?

(b) Study whether the behaviour of each of the constant solutions of the differential equation $y'(t) = f(y(t))$ is stable, unstable or semistable.

(c) Discuss the long-term behaviour of *all* solutions $y(t)$ to this differential equation in terms of their initial condition $y(0)$.

(d) Consider also the autonomous differential equation:

$$y'(t) = f(y(t)) + 0.1.$$

Compare the behavior of its solutions with the behavior of the solutions of $y'(t) = f(y(t))$. In particular, find how many constant solutions does the differential equation $y'(t) = f(y(t)) + 0.1$ have.

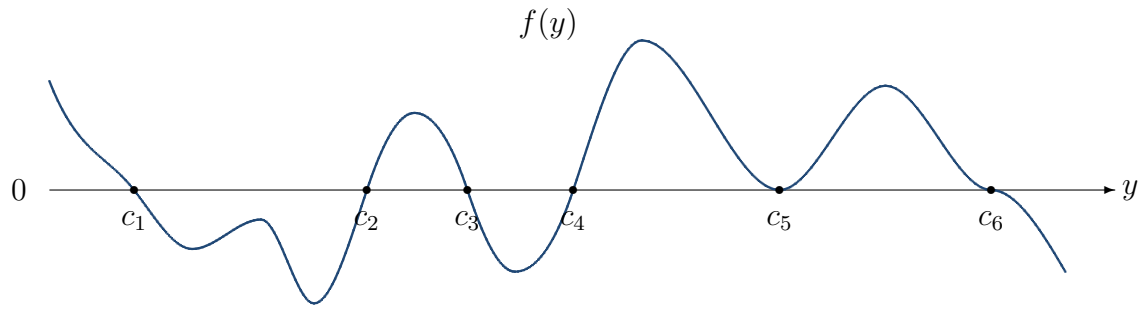
Problem Poblem 5 Solution: Read the graph of $f(y)$ (20 points)

In the supplied graph, the horizontal coordinate is y and the vertical coordinate is $f(y) = y'$. A graph above the horizontal axis means y increases; a graph below it means y decreases. The marked heights 3 and -1 are values of $f(y)$, not locations of constant solutions.

Name the zeros from left to right as

$$c_1 < c_2 < c_3 < c_4 < c_5 < c_6.$$

The graph gives their order, but not numerical values. The sketch below preserves the crossings and the touch at c_5; its distances are schematic.



(a) How many constant solutions are there? (3 points)

A constant solution $y(t) = c$ has $y' = 0$, so we need $f(c) = 0$ (1 point).

Count every point where the graph crosses *or touches* the horizontal axis. There are six (2 points):

$$\boxed{y(t) = c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6.}$$

The touch at c_5 counts even though the graph stays above the axis on both sides. The nearly flat crossing at c_6 also counts.

(b) Classify each constant solution (6 points)

Read the sign of f on each interval:

y	$(-\infty, c_1)$	(c_1, c_2)	(c_2, c_3)	(c_3, c_4)	(c_4, c_5)	(c_5, c_6)	(c_6, ∞)
$f(y)$	$+$	$-$	$+$	$-$	$+$	$+$	$-$
movement	$\rightarrow$	$\leftarrow$	$\rightarrow$	$\leftarrow$	$\rightarrow$	$\rightarrow$	$\leftarrow$

Compare the movement immediately below and above each zero:

Constant	From below	From above	Classification
c_1	toward	toward	stable (1 point)
c_2	away	away	unstable (1 point)
c_3	toward	toward	stable (1 point)
c_4	away	away	unstable (1 point)
c_5	toward	away	semistable (1 point)
c_6	toward	toward	stable (1 point)

(c) Long-term behavior for every initial value (6 points)

Use the smooth-curve interpretation of the supplied graph, with the end signs shown. As in Problem 2, a solution cannot cross a constant solution. On an interval with no zero of f, its direction cannot change (1 point).

Let $y_0 = y(0)$. Start with an example: if $c_1 < y_0 < c_2$, then the graph is below the axis, so $f(y_0) < 0$ and y decreases. It heads toward c_1. It cannot cross c_1, and it cannot settle at a point between c_1 and c_2 because the slope there is negative. Therefore its limit is c_1.

Apply the same steps to each interval. Include initial values exactly at a zero, which give constant solutions:

Initial value	Direction of $y(t)$	Limit as $t \to \infty$
$y_0 < c_1$	increases	c_1 (0.5 points)
$c_1 < y_0 < c_2$	decreases	c_1 (0.5 points)
$c_2 < y_0 < c_3$	increases	c_3 (0.5 points)
$c_3 < y_0 < c_4$	decreases	c_3 (0.5 points)
$c_4 < y_0 < c_5$	increases	c_5 (1 point)
$c_5 < y_0 < c_6$	increases	c_6 (0.5 points)
$y_0 > c_6$	decreases	c_6 (0.5 points)
$y_0 = c_i$, $i = 1, \ldots, 6$	stays constant	c_i (1 point)

Every solution is bounded between its starting value and the constant solution it approaches, so under this smooth-curve interpretation it exists for all future time.

At c_5, both arrows point toward larger y: this brings values below c_5 closer to it and takes values above c_5 farther away. Starting exactly at c_5 gives the constant solution $y(t) = c_5$. At c_6, the signs change from $+$ to $-$, so it is stable even though the curve is flat near the crossing.

(d) Replace the equation by $y' = f(y) + 0.1$ (5 points)

Write $g(y) = f(y) + 0.1$. Adding 0.1 shifts the entire graph upward by 0.1 (1 point). To find the new constant solutions, set $g(y) = 0$ and solve (1 point):

$$\begin{aligned} g(y) &= 0, \\ f(y) + 0.1 &= 0, \\ f(y) &= -0.1. \end{aligned}$$

Thus count where the *original* graph meets the horizontal line at height -0.1.

The small local maximum on the left has height -1. After shifting, its height is

$$-1 + 0.1 = -1 + \frac{1}{10} = -\frac{10}{10} + \frac{1}{10} = -\frac{9}{10} = -0.9 < 0.$$

It remains below the axis, so that small bump creates no new zeros. The larger negative dips also remain below the axis. At the touching point c_5,

$$g(c_5) = f(c_5) + 0.1 = 0 + 0.1 = 0.1 > 0,$$

so that zero disappears. Each of the other five crossings persists, at a shifted location (1 point).

Call the new zeros $b_1 < b_2 < b_3 < b_4 < b_5$. Reading the intersections with $f(y) = -0.1$ gives their order relative to the old zeros:

$$c_1 < b_1 < b_2 < c_2 < c_3 < b_3 < b_4 < c_4 < c_5 < c_6 < b_5.$$

For example, the first crossing moves to the right of c_1, because immediately to the right of c_1 the old graph drops from 0 to -0.1. The next crossing moves to the left of c_2, because the old graph reaches -0.1 before it rises to 0.

The five crossings and their directions are

New zero	Location	Signs of g: left, right	Classification
b_1	just right of c_1	$+, -$	stable
b_2	just left of c_2	$-, +$	unstable
b_3	just right of c_3	$+, -$	stable
b_4	just left of c_4	$-, +$	unstable
b_5	right of c_6	$+, -$	stable

Therefore (1 point),

The shifted equation has five constant solutions.

Its movement pattern is

$$\longrightarrow \quad b_1 \quad \longleftarrow \quad b_2 \quad \longrightarrow \quad b_3 \quad \longleftarrow \quad b_4 \quad \longrightarrow \quad b_5 \quad \longleftarrow .$$

This gives the complete long-term comparison:

Initial value	Direction for the shifted equation	Limit
$y_0 < b_1$	increases	b_1
$b_1 < y_0 < b_2$	decreases	b_1
$b_2 < y_0 < b_3$	increases	b_3
$b_3 < y_0 < b_4$	decreases	b_3
$b_4 < y_0 < b_5$	increases	b_5
$y_0 > b_5$	decreases	b_5
$y_0 = b_i$, $i = 1, \ldots, 5$	stays constant	b_i

The main change is that c_5 no longer stops an increasing solution. For any starting value $c_4 < y_0 \leq c_5$, the original solution approaches or stays at c_5; the shifted solution moves through that height and approaches b_5 (1 point). Starting at any old zero c_i now gives the positive initial slope $g(c_i) = 0.1$, rather than a constant solution.

The change is not simply an upward shift of every solution curve. It changes the slopes, the constant-solution heights, and which constant solution a given starting value approaches.

Problem 6. (20 pts) The Steller's sea cow (*Hydrodamalis gigas*) was a sirenian discovered in 1741. Let us suppose that the population $y(t)$ of Steller's sea cow, in the thousands unit, in the year $1741 + t$ naturally followed the logistic growth:

$$y'(t) = 3y(t)(1 - y(t)/5).$$

(a) Show that the long-term behavior of *any* solution of the above differential equation is 5, i.e. the population of Steller's sea cow would converge to five thousand.

In fact, excessive hunting caused the extinction of the Steller's sea cow in 1768. Let us study this scientifically: we choose to model hunting in our differential equation by adding a term $-H$ to the differential equation, where $H \in \mathbb{R}^+$ is a positive constant modeling the rate of hunting:

$$y'(t) = 3y(t)(1 - y(t)/5) - H.$$

(b) Knowing that the species eventually extincted, deduce that $H > 3.75$.

(c) Suppose that $y(0) = 1000$, i.e. there were a million sea cows in 1741. Find the exact rate of hunting which led to their extinction in 1768, i.e. find the exact value of H such that $y(27) = 0$.

Solution to Problem 6.

(a) We have

$$y' = 3y\left(1 - \frac{y}{5}\right).$$

Method 1: Separation of variables.
Separating variables and integrating,

$$\int \frac{5}{y(5-y)}\,dy = \int 3\,dt,$$

$$\ln\left|\frac{y}{5-y}\right| = 3t + C.$$

Solving for y gives

$$y(t) = \frac{5}{1 + Ce^{-3t}}.$$

For any positive initial population, since $e^{-3t} \to 0$ as $t \to \infty$,

$$\boxed{\lim_{t\to\infty} y(t) = 5.}$$

Method 2: Equilibrium and stability analysis.
Setting $y' = 0$, we find the equilibrium points:

$$3y\left(1 - \frac{y}{5}\right) = 0 \quad \Longrightarrow \quad y = 0,\ 5.$$

Examining the sign of y',

$$\begin{cases} y' > 0, & 0 < y < 5, \\ y' < 0, & y > 5. \end{cases}$$

Thus, solutions with $0 < y(0) < 5$ increase toward 5, while solutions with $y(0) > 5$ decrease toward 5.

Therefore, $y = 5$ is a stable equilibrium, and for any positive initial population,

$$\boxed{\lim_{t\to\infty} y(t) = 5.}$$

Hence, the population approaches 5000 sea cows.

(b) With hunting, the equation becomes

$$y' = 3y\left(1 - \frac{y}{5}\right) - H.$$

The natural growth rate is

$$g(y) = 3y - \frac{3}{5}y^2.$$

To find its maximum,

$$g'(y) = 3 - \frac{6}{5}y = 0 \quad \Longrightarrow \quad y = \frac{5}{2}.$$

Therefore,

$$g_{\max} = g\left(\frac{5}{2}\right) = \frac{15}{4}.$$

For a population initially above 5/2, as in part (c), extinction requires the hunting rate to exceed the maximum natural growth rate. Hence,

$$\boxed{H > \frac{15}{4} = 3.75.}$$

(c) We have $y(0) = 1000$ and $y(27) = 0$.

The equation is

$$y' = 3y - \frac{3}{5}y^2 - H.$$

Completing the square,

$$y' = -\frac{3}{5}\left(y - \frac{5}{2}\right)^2 - \left(H - \frac{15}{4}\right).$$

Let

$$a^2 = \frac{5}{3}\left(H - \frac{15}{4}\right).$$

Then

$$\frac{dy}{dt} = -\frac{3}{5}\left[\left(y - \frac{5}{2}\right)^2 + a^2\right].$$

Separating variables and using the initial and final conditions,

$$\int_{1000}^{0} \frac{dy}{(y - \frac{5}{2})^2 + a^2} = -\frac{3}{5}\int_0^{27} dt.$$

Integrating,

$$\frac{1}{a}\left[\tan^{-1}\left(\frac{y - \frac{5}{2}}{a}\right)\right]_{1000}^{0} = -\frac{81}{5}.$$

Thus,

$$\tan^{-1}\left(\frac{997.5}{a}\right) + \tan^{-1}\left(\frac{2.5}{a}\right) = \frac{81a}{5}.$$

This equation determines a exactly. Solving it numerically gives

$$a \approx 0.1892498213.$$

From the definition of a,

$$H = \frac{15}{4} + \frac{3}{5}a^2.$$

Substituting,

$$\boxed{H \approx 3.7714893.}$$

Thus, the hunting rate was approximately 3771.49 sea cows per year.

Problem 7. (20 pts) Consider the following initial value problem:

$$y'(t) = \sin(y(t) + t) - e^t, \quad y(0) = 4,$$

(a) Perform two steps of Euler's method with step $h = 0.1$ to obtain numerical approximations of $y(0.1)$ and $y(0.2)$.

(b) Give a numerical approximation to the value of the solution of the initial value problem at time $t = 0.3$, i.e. numerically approximate $y(0.3)$.

(c) Find an upper bound for the error in your approximation in Part (b).

Solution to Problem 7.

Let

$$f(t, y) = \sin(y + t) - e^t.$$

Euler's method is given by

$$y_{n+1} = y_n + hf(t_n, y_n),$$

with $h = 0.1$, $t_0 = 0$, and $y_0 = 4$.

(a) For the first step,

$$\begin{aligned} y_1 &= y_0 + hf(t_0, y_0) \\ &= 4 + 0.1(\sin(4) - 1). \end{aligned}$$

Hence,

$$\boxed{y(0.1) \approx 3.824320.}$$

For the second step,

$$\begin{aligned} y_2 &= y_1 + hf(t_1, y_1) \\ &= 3.824320 + 0.1\left[\sin(3.824320 + 0.1) - e^{0.1}\right]. \end{aligned}$$

Therefore,

$$\boxed{y(0.2) \approx 3.643281.}$$

(b) Using one more Euler step,

$$\begin{aligned} y_3 &= y_2 + hf(t_2, y_2) \\ &= 3.643281 + 0.1\left[\sin(3.643281 + 0.2) - e^{0.2}\right]. \end{aligned}$$

Thus,

$$\boxed{y(0.3) \approx 3.456590.}$$

(c) The error bound for Euler's method is

$$|y(t_n) - y_n| \leq \frac{hM}{2L}\left(e^{L(t_n - t_0)} - 1\right),$$

where L bounds $|f_y|$ and M bounds $|y''(t)|$ on $[0, 0.3]$.

First,

$$f_y(t, y) = \cos(y + t).$$

Since $|\cos(y + t)| \leq 1$, we take $L = 1$.

Differentiating the differential equation,

$$y'' = \cos(y + t)(y' + 1) - e^t.$$

Therefore,

$$|y''| \leq |y'| + 1 + e^t.$$

Also,

$$|y'| = |\sin(y + t) - e^t| \leq 1 + e^t.$$

Combining these inequalities,

$$|y''| \leq 2 + 2e^t.$$

On $[0, 0.3]$, we can take

$$M = 2 + 2e^{0.3} \approx 4.699718.$$

Substituting $h = 0.1$, $L = 1$, $M = 4.699718$, and $t_3 = 0.3$ into the error formula,

$$|y(0.3) - y_3| \leq \frac{0.1(4.699718)}{2}(e^{0.3} - 1).$$

Hence,

$$\boxed{|y(0.3) - y_3| \leq 0.082212.}$$