University of California Davis **Name (Print):** _____
Differential Equations MAT 22B **Student ID (Print):** _____

Sample Midterm II Examination **November 20 2026**
Time Limit: 50 Minutes

This examination document contains 10 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following second-order differential equation:

$$y''(t) + 9y(t) = 100 \cdot \delta(t-\pi) - 8000 \cdot u_{2\pi}(t).$$

(a) (10 points) Find *all* solutions to the differential equation.

Solution: Write $F(s) = \mathcal{L}[y(t)](s)$. Taking the Laplace transform of both sides implies

$$s^2F(s) - sy(0) - y'(0) + 9F(s) = 100 \cdot e^{-\pi s} - 8000 \cdot \frac{e^{-2\pi s}}{s},$$
$$\implies F(s) = 100 \cdot \frac{e^{-\pi s}}{s^2+9} - \frac{8000}{9} \cdot \left(\frac{e^{-2\pi s}}{s} - e^{-2\pi s} \frac{s}{s^2+9} \right) + \frac{s}{s^2+9} \cdot y(0) + \frac{y'(0)}{s^2+9},$$

where we used partial fractions for $e^{-2\pi s}/(s(s^2+9))$. Now, taking the inverse Laplace transform,

$$y(t) = \frac{100}{3} \cdot u_\pi(t) \sin(3(t-\pi)) - \frac{8000}{9} \cdot u_{2\pi}(t) (1 - \cos(3(t-2\pi))) + C_1 \cos(3t) + C_2 \sin(3t)$$

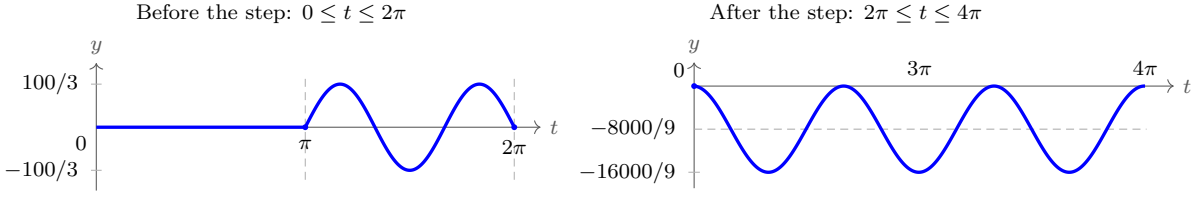
(b) (5 points) Find *all* solutions that satisfy $y(0) = 0$ and $y'(0) = 0$.

Solution: Before the first impulse at $t = \pi$, $y = C_1 \cos(3t) + C_2 \sin(3t)$ so the initial conditions give $C_1 = C_2 = 0$. Hence, there is exactly one solution:

$$\boxed{y(t) = \frac{100}{3} u_\pi(t) \sin\big(3(t-\pi)\big) - \frac{8000}{9} u_{2\pi}(t) \big[1 - \cos\big(3(t-2\pi)\big)\big].}$$

(c) (5 points) Qualitatively plot the solution in Part (b).
(Note that you do not need to have solved Part (b) for that.)

Solution: The solution is zero until $t = \pi$, then oscillates with amplitude $100/3$ until $t = 2\pi$. For $t \geq 2\pi$, we have $y = -8000/9 + (8000/9)\cos(3(t-2\pi)) - (100/3)\sin(3(t-2\pi))$. Thus, the later oscillations are centered at $-8000/9$ and have amplitude $\sqrt{(8000/9)^2 + (100/3)^2} \approx 889.5$.



2. (20 points) Consider the following second-order differential equation:

$$y''(t) + y(t) = u_1(t) - u_{1+2\pi}(t), \quad t \geq 0.$$

(a) (5 points) Consider the unique solution $y(t)$ such that $y(0) = 0$ and $y'(0) = 0$. Show that $y(t)$ vanishes, i.e. $y(t) = 0$ if $t \geq 1 + 2\pi$. Is $y(t) = 0$ for all $t \geq 0$?

Solution: Write $F(s) = \mathcal{L}[y(t)](s)$. Taking the Laplace transform of both sides implies

$$F(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-(1+2\pi)s}}{s(s^2+1)} = e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-(1+2\pi)s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right).$$

Hence, taking the inverse Laplace transform,

$$y(t) = u_1(t)\big(1 - \cos(t-1)\big) - u_{1+2\pi}(t)\big(1 - \cos(t-1-2\pi)\big), \quad t \geq 0,$$

is the unique solution with $y(0) = 0$ and $y'(0) = 0$. For $t \geq 1 + 2\pi$, both Heaviside functions equal 1. Since cosine is 2π-periodic, the two terms cancel. Therefore $y(t) = 0$ for all $t \geq 1 + 2\pi$. For $1 < t < 1 + 2\pi$, we have

$$y(t) = 1 - \cos(t-1),$$

which is positive. So, $y(t)$ is not identically 0 on $[0, \infty)$.

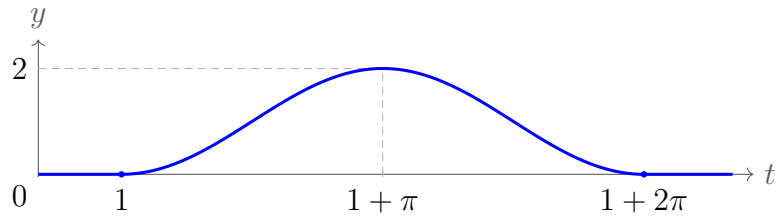
(b) (10 points) Explicitly find a formula for such $y(t)$.

Solution: From Part (a),

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 1 - \cos(t-1), & 1 \leq t \leq 1 + 2\pi, \\ 0, & t > 1 + 2\pi. \end{cases}$$

(c) (5 points) Justify whether $y(t)$ is a continuous or a discontinuous function and qualitatively plot the graph of $y(t)$ for $t \geq 0$.

Solution: Note that $y(1) = 0$ and $y(1 + 2\pi) = 0$, and $\lim_{t\to 1} y(t) = 0$ together with $\lim_{t\to 1+2\pi} y(t) = 0$. So, $y(t)$ is continuous.



3. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(a) (10 points) Find all solutions to the system above.

Solution: Let A be the given matrix and $\vec{b} = (0,1,0)^T$, where T denotes transpose. Since the forcing is constant, we look for a constant particular solution $\vec{v}_p$ satisfying $A\vec{v}_p + \vec{b} = 0$. Consider the ansatz $\vec{v}_p = (a_1, a_2, a_3)^T$. Plugging into the equation and solving for a_1, a_2, and a_3, we get a particular solution $\vec{v}_p = (-1/6, 0, 1/6)^T$. For the homogeneous system,

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{pmatrix} \\ &= (1-\lambda)(1+\lambda)^2 - 12 - (\lambda+1) + 12(\lambda+1) \\ &= -\lambda^3 - \lambda^2 + \lambda + 1 - 12 - \lambda - 1 + 12\lambda + 12 = -\lambda(\lambda-3)(\lambda+4)\,. \end{aligned}$$

Solving $(A - \lambda I)\vec{v} = 0$ gives the following eigenvectors:

$$\lambda_1 = 0 \,:\, \vec{v}_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, \qquad \lambda_2 = 3 \,:\, \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \qquad \lambda_3 = -4 \,:\, \vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

The eigenvalues are distinct, so these vectors are linearly independent. Adding the particular solution to the homogeneous solution gives

$$\boxed{\vec{x}(t) = \vec{v}_p + C_1\,\vec{v}_1 + C_2\,e^{3t}\,\vec{v}_2 + C_3\,e^{-4t}\,\vec{v}_3\,,}$$

with arbitrary real constants C_1, C_2, C_3.

(b) (5 points) Find the unique solution such that

$$x_1(4) = 1, \quad x_2(4) = 6, \quad x_3(4) = -13.$$

Solution: Since C_1, C_2, and C_3 are arbitrary, we can write the solution in Part (a) as

$$\vec{x}(t) = \vec{v}_p + C_1\,\vec{v}_1 + C_2\,e^{3(t-4)}\,\vec{v}_2 + C_3\,e^{-4(t-4)}\,\vec{v}_3\,.$$

Therefore, at $t = 4$, the initial conditions give

$$\vec{x}(4) = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Adding first and third components give $-12C_1 = -12$ and hence, $C_1 = 1$. The remaining equations are $2C_2 + C_3 = \frac{1}{6}$ and $3C_2 - 2C_3 = 0$. Solving these, we obtain $C_2 = \frac{1}{21}$ and $C_3 = \frac{1}{14}$. Hence, the unique desired solution is

$$\boxed{\vec{x}(t) = \vec{v}_p + \vec{v}_1 + \frac{1}{21}\, e^{3(t-4)}\, \vec{v}_2 + \frac{1}{14}\, e^{-4(t-4)}\, \vec{v}_3\,,}$$

(c) (2 points) Are there any constant solutions to this system ?

Solution: Yes. A constant solution $\vec{x}(t) = \vec{q} = (q_1, q_2, q_3)^T$ must satisfy $A\vec{q} + \vec{b} = 0$, which implies

$$q_1 + 2q_2 + q_3 = 0\,, \qquad 6q_1 - q_2 + 1 = 0\,.$$

We do not need the equation from the third row since it repeats the equation from the first row with the opposite sign. Then,

$$q_2 = 6q_1 + 1\,, \qquad q_3 = -q_1 - 2q_2 = -13q_1 - 2\,.$$

Therefore, all constant solutions are of the form

$$\boxed{\vec{x}(t) = \begin{pmatrix} q_1 \\ 6q_1 + 1 \\ -13q_1 - 2 \end{pmatrix}\,, \qquad q_1 \in \mathbb{R}\,.}$$

(d) (3 points) Compute the long-term behavior of the unique solution such that

$$x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = 0.$$

Substituting $\vec{x}(0) = (1, 0, 0)^T$ into Part (a) implies

$$C_1 = -\frac{1}{12}\,, \qquad C_2 = \frac{3}{7}\,, \qquad C_3 = \frac{11}{28}\,.$$

For example, adding the first and third component equations gives $-12C_1 = 1$; the other two then give $2C_2 + C_3 = 5/4$ and $3C_2 - 2C_3 = 1/2$. Thus,

$$\vec{x}(t) = \vec{v}_p - \frac{1}{12}\vec{v}_1 + \frac{3}{7}e^{3t}\vec{v}_2 + \frac{11}{28}e^{-4t}\vec{v}_3\,.$$

Hence,

$$\boxed{x_1(t) \longrightarrow +\infty\,, \qquad x_2(t) \longrightarrow +\infty\,, \qquad x_3(t) \longrightarrow -\infty\,.}$$

because e^{3t} term dominates as $t \to \infty$:

$$\lim_{t\to\infty} e^{-3t}\vec{x}(t) = \frac{3}{7}\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}\,.$$

4. (20 points) Let $\alpha \in \mathbb{R}$ and consider the following system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(a) (4 points) Find the interval of values for $\alpha \in \mathbb{R}$ such that the phase-portrait for the linear system above consists of a spiraling behavior[1].

Solution: The characteristic equation of $A = \begin{pmatrix} \alpha & 1 \\ -1 & 0 \end{pmatrix}$ is

$$\det(A - \lambda I) = \det \begin{pmatrix} \alpha - \lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 - \alpha\lambda + 1 = 0\,.$$

Hence, the eigenvalues are

$$\lambda_\pm = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}\,.$$

The eigenvalues $\lambda_\pm$ are nonreal exactly when $\alpha^2 - 4 < 0$. Therefore, the interval of spiraling is

$$\boxed{-2 < \alpha < 2\,.}$$

For $-2 < \alpha < 0$ the trajectories spiral inward; for $0 < \alpha < 2$ they spiral outward. At $\alpha = 0$, they are circles. The endpoints $\alpha = \pm 2$ have repeated real eigenvalues, in which case trajectories do not spiral.

(b) (4 points) For which values of $\alpha \in \mathbb{R}$ does *every* solution to the above linear system have long-term behavior equal to zero ?

Solution: Every solution tends to zero exactly when both eigenvalues have negative real part. If $|\alpha| < 2$, common real part of $\lambda_\pm$ is $\alpha/2$. If $|\alpha| \geq 2$, both eigenvalues are real and satisfy

$$\lambda_+ + \lambda_- = \alpha\,, \qquad \lambda_+ \cdot \lambda_- = 1\,.$$

Thus, both real eigenvalues are negative precisely when $\alpha < 0$. At the repeated-root case $\alpha = -2$, both e^{-t} and te^{-t} tend to zero. Combining these cases, we see that

$$\boxed{\alpha \in (-\infty, 0)}$$

implies a long-term behavior that is equal to 0.

[1] Concentric circles are also considered spirals.

(c) (4 points) Describe the long-term behavior of the unique solution $\vec{x}(t)$ to the linear system above such that $\vec{x}(0) = \begin{pmatrix} -15 \\ 3 \end{pmatrix}$ for the value $\alpha = -2$.

Solution: For $\alpha = -2$,

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \det(A - \lambda I) = (\lambda + 1)^2 .$$

The eigenvalue $\lambda = -1$ is repeated. To find an eigenvector, we solve

$$(A + I)\vec{v} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 .$$

Thus $v_1 = v_2$, and we may take $\vec{v} = (1, 1)^T$. There is only one linearly independent eigenvector, so we look for a generalized eigenvector $\vec{w}$ satisfying

$$(A + I)\vec{w} = \vec{v}, \quad \text{which implies} \quad -w_1 + w_2 = 1 .$$

Take $\vec{w} = (0, 1)^T$, which satisfies $(A + I)\vec{w} = \vec{v}$. The two independent solutions are

$$e^{-t}\vec{v} = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad e^{-t}(t\vec{v} + \vec{w}) = e^{-t} \begin{pmatrix} t \\ t + 1 \end{pmatrix} .$$

Hence, the general solution is

$$\vec{x}(t) = e^{-t} \left[C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} t \\ t + 1 \end{pmatrix} \right] .$$

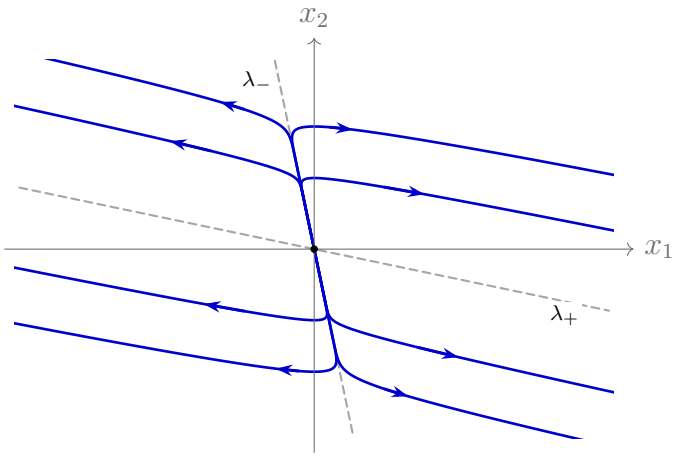
The initial condition gives $C_1 = -15$ and $C_1 + C_2 = 3$, so $C_2 = 18$. Therefore,

$$\boxed{\vec{x}(t) = e^{-t} \begin{pmatrix} -15 + 18t \\ 3 + 18t \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{as } t \to \infty .}$$

Both components tend to zero because $e^{-t} \to 0$ and $te^{-t} \to 0$.

(d) (4 points) Plot qualitatively the phase-portrait of the system for $\alpha = 5$.

Solution: The eigenvalues are $\lambda_\pm = (5 \pm \sqrt{21})/2$, with $0 < \lambda_- < \lambda_+$. Hence the origin is an unstable node. The second row of eigenvalue equation $A\vec{v} = \lambda$ gives $\vec{v}$ $-v_1 = \lambda v_2$, so we may take $\vec{v}_\pm = (\lambda_\pm, -1)^T$; whose directions are the dashed lines in the plot below. The curved trajectories approach the λ_+ direction as $t \to \infty$.

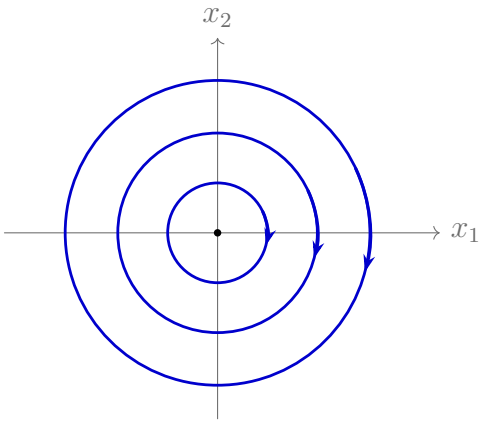


(e) (4 points) Plot qualitatively the phase-portrait of the system for $\alpha = 0$.

Solution: Now $x_1' = x_2$ and $x_2' = -x_1$. So,

$$\frac{d}{dt}(x_1^2 + x_2^2) = 2x_1'x_1 + 2x_2'x_2 = 2x_1x_2 - 2x_2x_1 = 0\,.$$

Therefore, $x_1^2 + x_2^2 = r^2$ is constant for some $r \geq 0$. For $r > 0$ every trajectory is a circle about the origin, while for $r = 0$, we have the origin as an equilibrium point. At $(r, 0)$ the velocity is $(0, -r)$, so the motion is *clockwise*.



5. (20 points) For each of the ten sentences below, circle whether they are **true** or **false**.

(a) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(1) True. (2) False.

Solution: **True.** The matrix satisfies $A^2 = 0$. So, by Taylor series,

$$e^A = I + A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

(b) (2 points) The exponential of the matrix $\begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}$ is the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(1) True. (2) False.

Solution: **True.** Since $A^2 = -(2\pi)^2 I$, we have $e^A = \cos(2\pi)I + \frac{\sin(2\pi)}{2\pi}A = I$.

(c) (2 points) Any solution of $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is of the form $Ce^{2t}\xi_2$.

(1) True. (2) False.

Solution: **True** Every solution is of the form $e^{2t}\vec{x}(0)$. (True if by ξ_2 it is meant an arbitrary vector that may be chosen for each solution, but False if ξ_2 is one fixed eigenvector.)

(d) (2 points) Any solution of $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is of the form $Ce^{2t}\xi_2$.

(1) True. (2) False.

Solution: **False.** The solution with $\vec{x}(0) = (0,1)^T$ is $e^{2t}(t,1)^T$.

(e) (2 points) The Laplace transform of e^{-t^2} does not exist.

(1) True. (2) False.

Solution: **False.** The integral $\int_0^\infty e^{-t^2-st}\,dt$ converges for every real s.

(f) (2 points) A linear system of differential equations $\vec{x}(t)' = A\vec{x}(t)$ with $\det(A) \neq 0$ does not have a non-zero constant solution.

(1) True. (2) False.

Solution: **True.** A constant solution must satisfy $\vec{x}' = \vec{0} = A\vec{x}$. Thus, A being invertible requires $\vec{x} = 0 = A^{-1}\vec{0}$.

(g) (2 points) A linear system $\vec{x}(t)' = A\vec{x}(t) + g(t)$ with $\det(A) \neq 0$ cannot have a non-zero constant solution even if $g(t)$ is constant.

(1) True. (2) False.

Solution: **False.** For a nonzero constant forcing $g(t) = \vec{b}$, the vector $\vec{x} = -A^{-1}\vec{b}$ is a nonzero constant solution.

(h) (2 points) The Laplace transform $\mathcal{L}(f)(s)$ of $f(t) = e^{t^2}$ is $\mathcal{L}(e^{t^2})(s) = s^{-2}$.

(1) True. (2) False.

Solution: **False.** The integral $\int_0^\infty e^{t^2 - st}\, dt$ diverges for every real s.

(i) (2 points) The vector $e^{At} \cdot x_0$ solves the Initial Value Problem $\vec{x}(t)' = A\vec{x}(t)$ with initial condition $x(0) = x_0$ if and only if A is diagonalizable.

(1) True. (2) False.

Solution: **False.** The formula $\vec{x}(t) = e^{At}\vec{x}_0$ works for every constant matrix A, diagonalizable or not.

(j) (2 points) The Laplace transform $\mathcal{L}(ty(t))$ equals minus the s-derivative of $\mathcal{L}(y(t))(s)$.

(1) True. (2) False.

Solution: **True.** See Part (c) of Problem 2 in Homework 4.