University of California Davis **Name (Print):** _____
Differential Equations MAT 22B **Student ID (Print):** _____

Sample Midterm II Examination 2 **November 20 2026**
Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider the following second-order differential equation:

$$\boxed{y''(t) + 2y'(t) + 2y(t) = \lambda \cdot \delta(t - 50)}$$

depending on a real value $\lambda \in \mathbb{R}$.

(a) (5 points) Qualitatively plot the solution $y(t)$ of the differential equation such that $y(0) = 10$, $y'(0) = 0$ for each of the five values $\lambda \in \{-200, -100, 0, 100, 200\}$.

(b) (10 points) Find *all* solutions to the differential equation for $\lambda = 100$.

(c) (5 points) Does the long term behavior $\lim_{t\to\infty} y(t)$ of the solutions with $y(0) = 10$, $y'(0) = 0$ of the original differential equation depend on λ? (Justify your answer.)

2. (20 points) Consider the following initial value problem:

$$\boxed{y^{(4)}(t) + 5y''(t) + 4y(t) = u_1(t) - u_3(t) + \delta(t-5)}$$

$$y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

(a) (10 points) Justify that the following two Laplace transforms are correct:

$$\mathcal{L}\left(\frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos(2t)\right) = \frac{1}{s(s^2+1)(s^2+4)}$$

$$\mathcal{L}\left(\frac{1}{3}\sin t - \frac{1}{6}\sin(2t)\right) = \frac{1}{(s^2+1)(s^2+4)}$$

(b) (10 points) Show that the function

$$y(t) = u_1(t)f(t-1) - u_3(t)f(t-3) + u_5(t)g(t-5).$$

is the unique solution to the initial value problem, where we denoted

$$f(t) := \frac{1}{4} - \frac{1}{3}\cos t + \frac{1}{12}\cos(2t),$$

$$g(t) := \frac{1}{3}\sin t - \frac{1}{6}\sin(2t).$$

3. (20 points) Consider the following linear system of differential equations:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} e^t \\ t \\ 0 \end{pmatrix}$$

(a) (10 points) Find two linearly independent solutions to **homogeneous** system.

(b) (5 points) Find all solutions to the **homogeneous** system.

(c) (5 points) Find all solutions to the original **non-homogeneous** system.

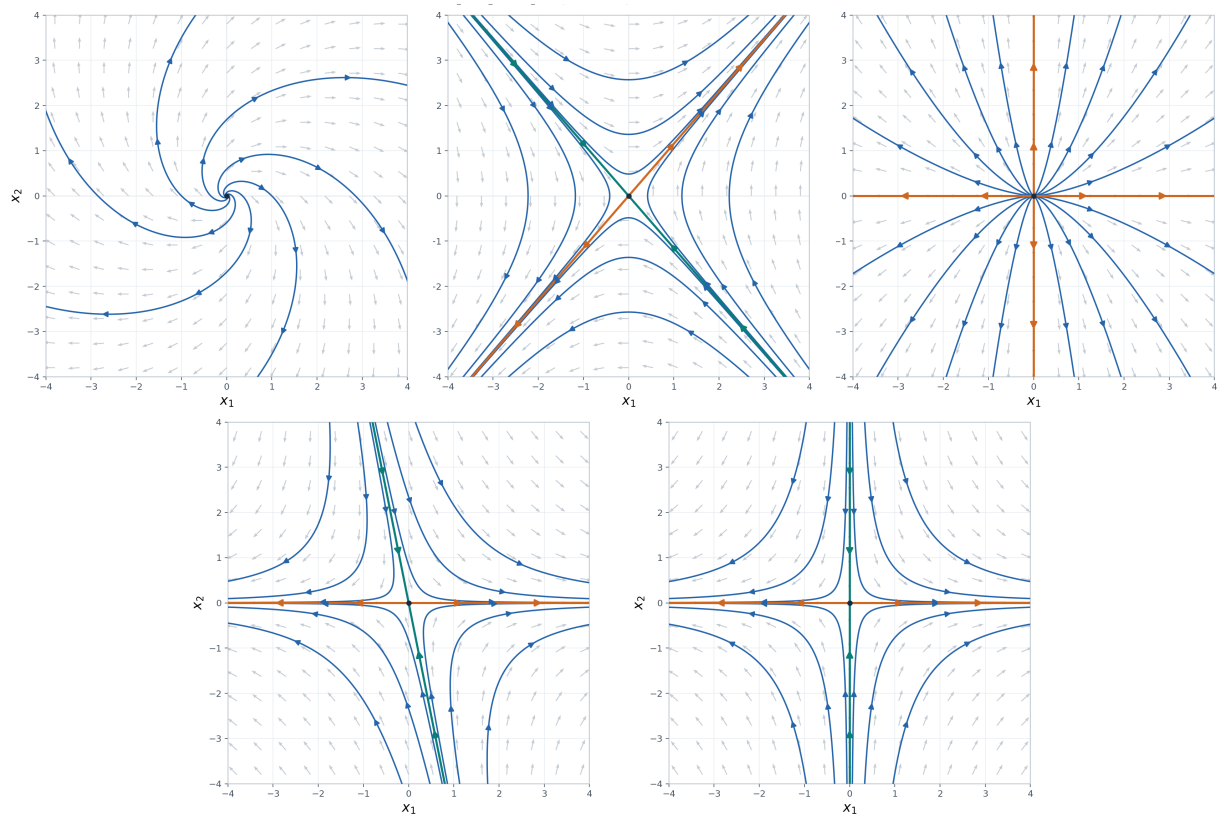


Figure 1: Phase portraits from Problem 4

4. (20 points) Consider a 2×2 system of differential equations $\dot{\vec{x}}(t) = A\vec{x}$ and the following 5 matrices A:

$$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

   (a) (10 points) Match each of the phase portraits in Figure 1 with the corresponding system $\dot{\vec{x}}(t) = A\vec{x}$ for each of the 5 matrices above. (Justify your answers.)

(b) (5 points) Suppose that we add the matrix

$$E_\varepsilon := \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$$

to each of the matrices A above, with $\varepsilon \in \mathbb{R}_{>0}$ arbitrary small. Explain how the phase portraits of $\dot{\vec{x}}(t) = (A + E_\varepsilon)\vec{x}$ qualitatively change with respect to those of $\dot{\vec{x}}(t) = A\vec{x}$. (Justify your answer.)

(c) (3 points) Consider the following system of differential equations

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Describe the long term behavior of $x_1(t)$ and $x_2(t)$.

(d) (2 points) Consider the following system of differential equations

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Describe the phase portraits for the different values of $\lambda \in [0, 4]$.

5. (20 points) For each of the five sentences below, circle the unique correct answer.

(a) (4 points) $c_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{2t} v_2$ solves $\begin{pmatrix}x_1'(t)\\x_2'(t)\end{pmatrix} = \begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$ if v_2 is

(1) $\begin{pmatrix}1\\0\end{pmatrix}$ (2) $\begin{pmatrix}0\\1\end{pmatrix}$ (3) $\begin{pmatrix}1\\t\end{pmatrix}$ (4) $\begin{pmatrix}t\\1\end{pmatrix}$

(b) (4 points) If $A = \begin{pmatrix}-4&1\\0&-4\end{pmatrix}$, the exponential e^{tA} is

(1) $e^{-4t}\begin{pmatrix}1&1\\0&1\end{pmatrix}$ (2) $e^{-4t}\begin{pmatrix}1&0\\0&1\end{pmatrix}$ (3) $e^{-4t}\begin{pmatrix}1&t\\0&1\end{pmatrix}$ (4) $e^{-4t}\begin{pmatrix}1&1+t\\0&1\end{pmatrix}$

(c) (4 points) A particular solution to $\begin{pmatrix}x_1'(t)\\x_2'(t)\end{pmatrix} = \begin{pmatrix}2&1\\0&2\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} + \begin{pmatrix}t\\e^t\end{pmatrix}$ is

(1) $\begin{pmatrix}e^{-t} - t/2 - 1/4\\-e^{-t}\end{pmatrix}$ (2) $\begin{pmatrix}e^{t} - t^2/2 - 1/4\\-e^{t}\end{pmatrix}$ (3) $\begin{pmatrix}e^{t} - t/2 - 1/4\\-e^{t}\end{pmatrix}$

(d) (4 points) Consider the solution $y(t)$ of $y''(t) + 16y(t) = \delta(t-1) + \delta(t-c)$, with $y(0) = y'(0) = 0$ and $c > 1$. Then, for all time $t > c$, $y(t)$ is

(1) Never zero for any $c > 1$. (2) Always zero for all $c > 1$.

(3) Zero only for finitely many $c > 1$. (4) Zero for infinitely many $c > 1$.

(e) (4 points) The Laplace transform of $e^{3t}\cos(4t)$ is:

(1) $\frac{s}{s^2+16}$ (2) $\frac{s}{(s-3)^2+16}$ (3) $\frac{s-3}{(s-3)^2+16}$ (4) $\frac{s-3}{s^2+16}$ (4) $\frac{s+3}{s^2+16}$