Kernels & images $\to$ range (6.2, 6.3)

$\searrow$ null space

Input $f: V \xrightarrow{\text{linear}} W$

$\quad\quad\quad\quad \cup \quad\quad\quad \cup$

output $\rightsquigarrow$ $\mathrm{Ker}(f)$ $\quad \mathrm{Im}(f)$

Example:

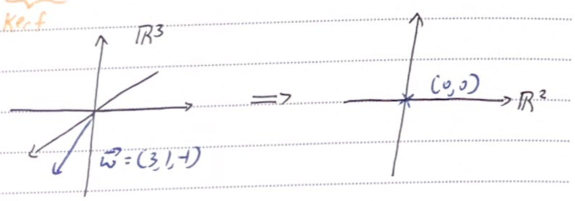
$f: \mathbb{R}^3 \to \mathbb{R}^2 \leftarrow \{(1,0), (0,1)\}$

in the axs basis $\{(1,0,0), (0,1,0), (0,0,1) \text{ in } \mathbb{R}^3\}$,

defined by $A_f = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 5 & 1 \end{pmatrix}$

$f(1,1,2) = \boxed{(7, 28)} \to$ non-zero $= (1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} + \begin{pmatrix} 6 \\ 22 \end{pmatrix}$

$\underbrace{f(3,1,-1)}_{\mathrm{Ker} f} = \boxed{(0,0)} \to$ zero



In V, vectors that survive $(f(v) \neq 0)$

are not a subspace

(e.g. $\vec{0}$ is not in it!)

Lemma: Ker f

Given $f: V \to W$ linear, then $\underbrace{\{v \in V : f(v) = 0\}}_{\mathrm{Ker}(f)} \subseteq V$

is a subset

Proof of lemma:

need to check closed under addition & scalar

multiplication,

① If $\underbrace{v_1, v_2 \in \mathrm{Ker}(f)}$, then (?) $\underbrace{v_1 + v_2 \in \mathrm{Ker} f}$

$f(v_1) = 0 \quad\quad\quad\quad f(v_1 + v_2) = 0\,?$

$f(v_2) = 0 \quad\quad\quad\quad = f(v_1) + f(v_2)$

$\quad\quad\quad\quad\quad\quad\quad\quad = 0 + 0 = 0$

② If $\underbrace{v \in \mathrm{Ker} f}$, then $\underbrace{a \cdot v \in \mathrm{Ker} f}$, $(a \in \mathbb{R})$

$\quad\quad\quad\quad\quad\quad\quad\quad f(a \cdot v) = 0\,?$

$f(v) = 0 \quad\quad\quad\quad = a \cdot f(v) = a \cdot 0 = 0$

Lemma: $Im(f)$

$$\boxed{\text{Let } f: V \to W \text{ be linear, then } \underbrace{\{w \in W : \exists v \in V \text{ with } f(v) = w\}}_{= im(f)} \subseteq W \text{ is a subspace}}$$

Proof of lemma:

$w_1, w_2 \in im(f) \subseteq W$, i.e. $\left.\begin{array}{l} \exists v_1 \text{ s.t. } f(v_1) = w_1 \\ \exists v_2 \text{ s.t. } f(v_2) = w_2 \end{array}\right\}$ given

$w_1 + w_2 \in im(f)$ → trying to prove

Need to build $v \in V$ s.t. $f(v) = w_1 + w_2$

$\Downarrow \Downarrow$

$f(v_1) + f(v_2) = f(v_1 + v_2)$

↳ linear

Take $v = v_1 + v_2$

Then $f(v_1 + v_2) = f(v_1) + f(v_2) = w_1 + w_2$

Scalar multiplication: $\boxed{\begin{array}{l} a \in \mathbb{R} \\ w \in Im(f) \\ (w = f(v)) \end{array}}$ $\Rightarrow$ $aw \in Im(f)$

given

want

Then, $v' = av$, $f(v') = f(av) = af(v) = aw$

Example:

$f: \underset{V}{\underset{\parallel}{\mathbb{R}^3}} \to \underset{W}{\underset{\parallel}{\mathbb{R}^3}}$, given by $A_f = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ -2 & 2 & 4 \end{pmatrix}$

(i) Find $Ker(f) \subseteq V$ & a basis for it.

(ii) Find $im(f) \subseteq W$ & a basis for it.

→ homogeneous linear system

(i) Ker f $\exists v = (x_1, x_2, x_3)$ s.t. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ -2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

~~the only sol^n is $Ker(f) = \{0\}$~~

the sol^n is $Ker(f) = \langle (1, -2, \frac{3}{2}) \rangle$ - dim 1

$= \langle (2, -4, 3) \rangle$

→ $(1, 7, -13)$

(ii) $im(f) \subseteq W = \mathbb{R}^3$, $w \in im(f)$ if $\exists (x_1, x_2, x_3) \in V$

s.t.

$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ -2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -13 \end{pmatrix}$ ← non-homogeneous linear system

↳ 1 particular image

→ Observe $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ are all in $im(f)$

$\dim im(f) \geq 2$, show $\dim im(f) = 2$, ie.

$\exists w \in \mathbb{R}^3$ s.t. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ -2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = w$

has no sol^n,