The dimension formula $\boxed{\mathbb{R}^3 \xrightarrow{\text{(project)}} \mathbb{R}^2}$ → example

$f: V \xrightarrow{\text{linear}} W$

$\cup\!| \qquad \cup\!|$

$Ker(f) \quad im(f)$

Theorem:

Let V be a finite dimensional v.s.
Consider $f: V \to W$ a linear map. Then $\dim(im f) < \infty$
and $\boxed{\dim(V)} = \boxed{\dim Ker(f)} + \cancel{im(f)}\ \boxed{\dim(im(f))}$

Intuition: f: Covid 19

$\dim(V)$ = population = people who died of Covid 19 + people who survived Covid 19

$= \dim Ker(f) + \dim(im(f))$

Proof of theorem: say write $n = \dim V$, $m = \dim Ker(f)$

Choose a basis of V as follows:

① choose a basis $\{v_1, \dots, v_m\}$ of $Ker(f)$

② Complete basis in ① to basis of V:

$\{\underbrace{v_1, v_2, \dots, v_m}_{\text{kernel}}, \overbrace{u_{m+1}, \dots, u_n}^{\text{some choice}}\}$

Consider the set of vectors in W:

$\{f(u_{m+1}), \dots, f(u_n)\} \subseteq im(f)$

we want to show $\{f(u_j)\}$ is a basis of $im(f)$

① spanning set for $im(f)$

② linear independence

① $w \in im(f)$

so $w = f(v)$

$v = [a_1 v_1 + \dots + a_m v_m] + a_{m+1} u_{m+1} + \dots + a_n u_n$

$w = f(v) = 0 + a_{m+1} f(u_{m+1}) + \dots + a_n f(u_n)$

② To show linear independence, suppose that $\{f(u_{m+1}), \dots, f(u_n)\}$ are not linear independent

$\exists b_i \in \mathbb{R}$, s.t. $b_{m+1} f(u_{m+1}) + \dots + b_n f(u_n) = 0$

Then, $f(b_{m+1} u_{m+1} + \dots + b_n u_n) = 0$ (linearity)

So, $b_{m+1} u_{m+1} + \dots + b_n u_n \in Ker(f)$. Thus, since $\{v_1, \dots, v_m\}$ basis of ker, $\exists c_1, \dots, c_m$ s.t.

$\boxed{b_{m+1} u_{m+1} + \dots + b_n u_n = c_1 v_1 + \dots + c_m v_m}$

$\Rightarrow \because \{v_1, \dots, v_m, u_{m+1}, \dots, u_n\}$ basis, this is a contradiction

Relation between Ker, im, dim & injectivity & surjectivity

Def: $f: V \xrightarrow{\text{Linear}} W$ is said to be

(i) injective if $\forall v$ s.t. $f(v)=0$ then $v=0$

$\Rightarrow$ if $v_1 \neq v_2$, $f(v_1) \neq f(v_2)$

(ii) surjective if $\forall w \in W \ \exists v \in V$ s.t. $f(v)=w$

(iii) bijective if injective & surjective

Example:

① $f: \mathbb{R}^3 \to \mathbb{R}^2$, $A_f = \begin{pmatrix} 3 & -6 & 6 \\ -1 & 2 & -2 \end{pmatrix}$

① It is not injective:

$v_1 = (2,1,0)$,

$v_2 = (0,0,0)$

but $f(v_1) = 0 = f(v_2)$

② Is it surjective?

Consider $v_3 = (4,1,-1)$ ~~ws~~, $v_3 \in Ker(f)$

So $\dim Ker(f) \geq 2$ b.c. v_1, v_3 are linearly independent

$\rightarrow =$ b.c. $f \neq 0$

$\Rightarrow \dim im f = 1$

NOT SURJECTIVE

Lemma

$f: V \to W$ Linear. Then:

① f injective $\Leftrightarrow$ $Ker f = \{0\}$

② f surjective $\Leftrightarrow$ $im f = W$ ($\Leftrightarrow \dim V - \dim Ker = \dim W$)

③ If ~~$\dim V \neq \dim W$~~ $\dim V \not\leq \dim W$, then f is not injective

④ If $\dim V \not\geq \dim W$, then f is not surjective