LECTURE 2: PRACTICE EXERCISES

MAT-67 SPRING 2024

ABSTRACT. These practice problems correspond to the 2nd lecture of MAT-67 Spring 2024, delivered on April 3rd 2024.

The following are practice problems. They are not to be submitted, they are for your own practice. Solutions will be posted soon.

Recall that a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be linear if it satisfies the following 2 conditions:

(i) $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$,
(ii) $f(c \cdot x) = c \cdot f(x)$, for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

See lecture notes from Lectures 1 & 2, and also Section 1.3 in book, for more details.

Problem 1. For each of the following maps, prove whether it is *linear* or *non-linear*.

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x$,
(2) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x + 1$,
(3) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \cos(x)$,
(4) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3 - x$,
(5) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \ln(1 + x^2)$,
(6) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x_1, x_2) = x_1 + 4x_2$,
(7) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x_1, x_2) = 3x_1 - x_2 + 7$,
(8) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (3x_1 - x_2, x_2)$,
(9) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 4x_1 + x_3)$,
(10) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 1)$
(11) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1x_3, x_1 - x_2)$
(12) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $f(x_1, x_2, x_3) = (e^{x_3+x_1}, 3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, 0)$

Problem 2. For each of the following pairs of maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$, write their composition $g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$, defined by

$$(g \circ f)(x_1, \dots, x_n) = g(f((x_1, \dots, x_n))).$$

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 3x$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(s) = 4s + 1$.

(2) $f : \mathbb{R} \longrightarrow \mathbb{R}^2$, $f(x) = (2x, 7x)$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g(s, t) = s + 6t$.

(3) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (2x + 3y, 7x - y)$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $g(s, t) = 3s - t$.

(4) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x, y) = (x - 2y, 4x + 7y, x)$, and the map $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $g(s, t, u) = (s + 3t - u, s + u)$.

Problem 3. **Prove**, with an argument, or **disprove**, with a counter-example, each of the statements sentences below.

(1) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are two maps. If f and g are linear, then the composition $g \circ f$ is linear.

(2) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are two maps. If f is linear, then the composition $g \circ f$ is linear.

(3) Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are two maps. If f is not linear, then the composition $g \circ f$ is never linear.

(4) For any map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ there exists a linear map $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ such that the composition $g \circ f$ is linear.

(5) For any non-linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ there exists a linear map $g : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ such that the composition $g \circ f$ is not linear.

Problem 4. Suppose that a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$.

(1) Show that $f(n \cdot x) = n \cdot f(x)$ for all natural numbers $n \in \mathbb{N}$.
(2) Show that $f(q \cdot x) = q \cdot f(x)$ for all rational numbers $q \in \mathbb{Q}$.

In particular, a continuous function satisfying condition (i) of linearity also satisfies condition (ii).

Problem 5. (*Hard, only for fun*) Solve the following parts:

(1) Find a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$f(x + y) = f(x) + f(y), \quad \text{for all } x, y \in \mathbb{R}$$

but it is *not* linear.

Hint: By Problem 4, this map f cannot be continuous.

(2) Find a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$f(c \cdot x) = c \cdot f(x), \quad \text{for all } c \in \mathbb{R}, x \in \mathbb{R}$$

but it is *not* linear.