LECTURE 3: PRACTICE EXERCISES

MAT-67 SPRING 2024

Abstract. These practice problems correspond to the 3rd lecture of MAT-67 Spring 2024, delivered on April 5th 2024. Solutions were typed by TA Scroggin, please contact *tmscroggin – at – ucdavis.edu* for any comments.

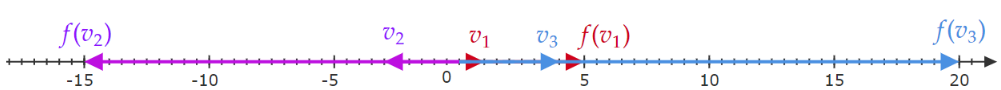
The following are practice problems. They are not to be submitted, they are for your own practice. Solutions will be posted soon.

Problem 1. Draw in the real line $\mathbb{R}$ and the real plane $\mathbb{R}^2$ the following maps f by drawing vectors v_i and their images $f(v_i)$.

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x$ and the vectors
$$v_1 = (1), \quad v_2 = (-3), \quad v_3 = (4)$$
(2) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (3x_1 - x_2, x_2)$ and the vectors
$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (2, 5)$$
(3) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (3x_1, 5x_2)$ and the vectors
$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (1, 1)$$
(4) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (x_1, x_1)$ and the vectors
$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (1, 1)$$
(5) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (2x_1, 0)$ and the vectors
$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (2, -3)$$
(6) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (-x_1, -x_2)$ and the vectors
$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (5, 6)$$

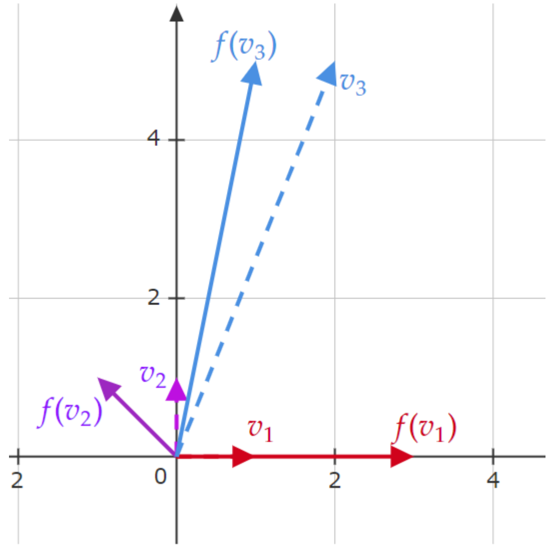
Solution.

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x$



Vectors:
$$v_1 = (1), \quad v_2 = (-3), \quad v_3 = (4)$$
$$f(v_1) = (5), \quad f(v_2) = (-15), \quad f(v_3) = (20)$$

(2) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (3x_1 - x_2, x_2)$

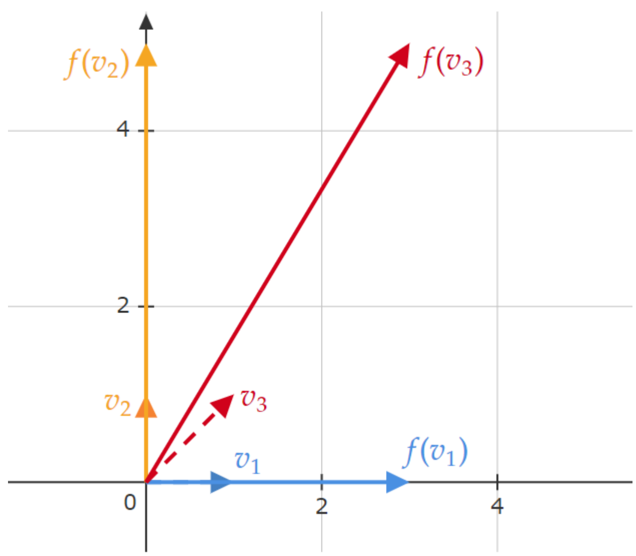


Vectors:

$$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (2,5)$$

$$f(v_1) = (3,0), \quad f(v_2) = (-1,1), \quad f(v_3) = (1,5)$$

(3) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (3x_1, 5x_2)$

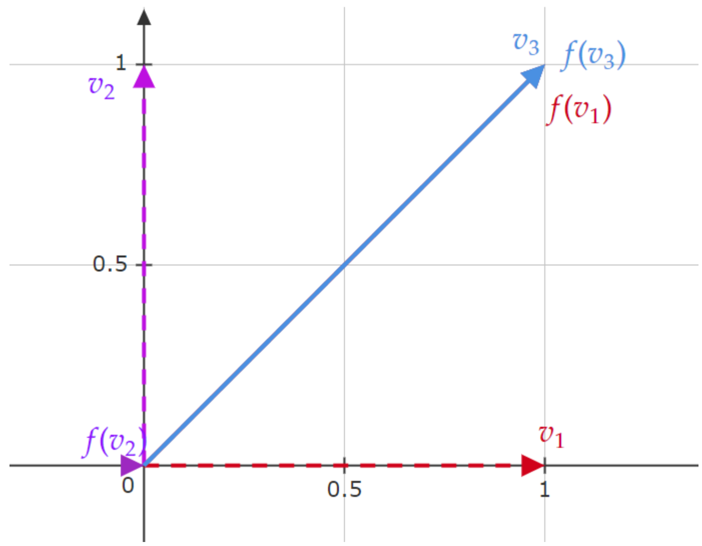


Vectors:

$$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (1,1)$$

$$f(v_1) = (3,0), \quad f(v_2) = (0,5), \quad f(v_3) = (3,5)$$

(4) $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1,x_2)=(x_1,x_1)$

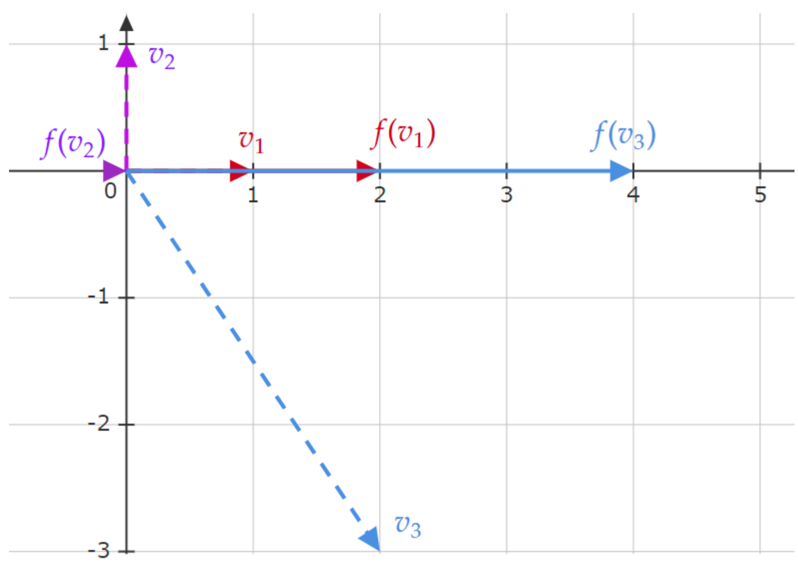


Vectors:

$$v_1=(1,0),\quad v_2=(0,1),\quad v_3=(1,1)$$

$$f(v_1)=(1,1),\quad f(v_2)=(0,0),\quad f(v_3)=(1,1)$$

(5) $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1,x_2)=(2x_1,0)$

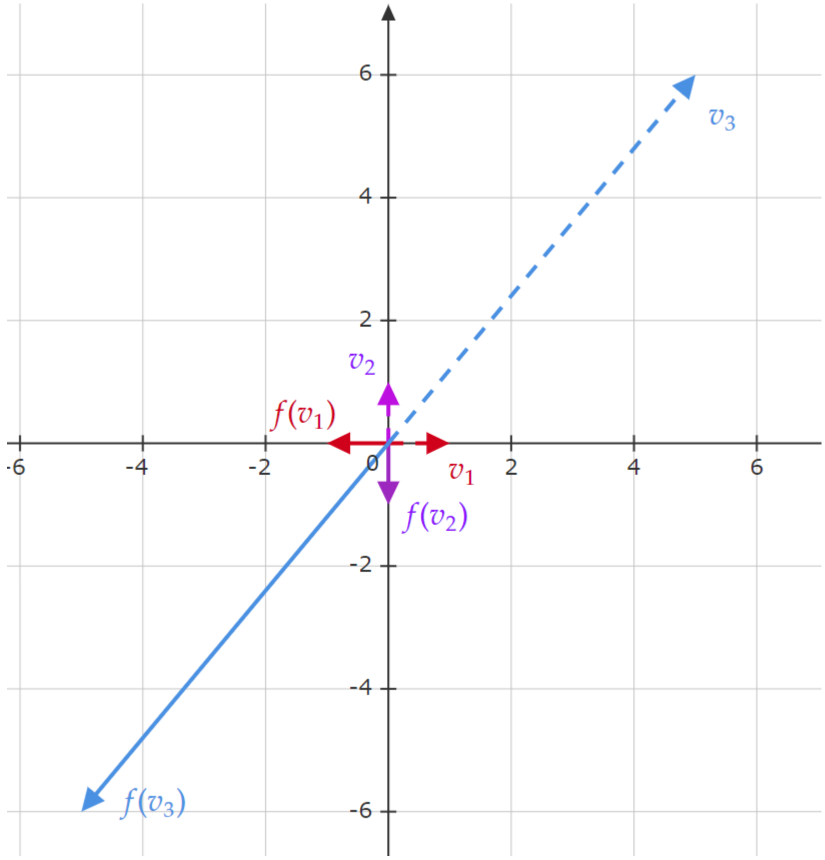


Vectors:

$$v_1=(1,0),\quad v_2=(0,1),\quad v_3=(2,-3)$$

$$f(v_1)=(2,0),\quad f(v_2)=(0,0),\quad f(v_3)=(4,0)$$

(6) $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1,x_2)=(-x_1,-x_2)$



Vectors:

$$v_1=(1,0), \quad v_2=(0,1), \quad v_3=(5,6)$$

$$f(v_1)=(-1,0), \quad f(v_2)=(0,-1), \quad f(v_3)=(-5,-6)$$

$\square$

Problem 2. Solve the following parts:

(1) Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a linear map such that $f(1)=3$. Compute $f(4)$.

(2) Suppose that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a linear map such that $f(7)=-2$. Compute $f(5)$.

(3) Suppose that $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is a linear map such that $f(1,0)=(3,4)$ and $f(0,1)=(0,2)$. Compute $f(-1,5)$.

(4) Suppose that $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is a linear map such that $f(2,0)=(3,1)$ and $f(0,4)=(0,-1)$. Compute $f(7,1)$.

(5) Suppose that $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is a linear map such that $f(1,1)=(1,2)$ and $f(2,3)=(-4,9)$. Compute $f(1,1)$.

(6) Suppose that $f:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is a linear map. Compute $f(0,0)$.

(7) Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a linear map such that $f(1,0) = 3$ and $f(0,1) = 2$. Compute $f(-1,5)$.

Solution.

(1) A linear map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $f(x) = ax$ for some $a \in \mathbb{R}$. By evaluation at $x = 1$ we see that

$$f(1) = a(1) = a.$$

Therefore, $a = 3$, so the map f is defined as $f(x) = 3x$. We compute $f(4) = 3(4) = \boxed{12}$.

(2) The linear map is of the form $f(x) = ax$. By evaluating at $x = 7$, we have

$$f(7) = a(7) = 7a.$$

We are given the initial condition that $f(7) = -2$, so we solve for a

$$7a = -2$$
$$a = -\frac{2}{7}$$

So the map is defined $f(x) = -\frac{2}{7}x$. And we compute

$$f(5) = -\frac{2}{7}(5) = \boxed{-\frac{10}{7}}.$$

(3) A linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is of the form $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ where a, b, c, d are real constants. We use the given function values to solve for a, b, c, d.

$$f(1,0) = (a(1) + b(0), c(1) + d(0)) = (a, c) = (3, 4),$$
$$f(0,1) = (a(0) + b(1), c(0) + d(1)) = (b, d) = (0, 2)$$

Therefore, $a = 3, b = 0, c = 4, d = 2$ and the function f is defined

$$f(x_1, x_2) = (3x_1, 4x_1 + 2x_2).$$

Now, we compute the value of the function at $(-1, 5)$

$$f(-1,5) = (3(-1), 4(-1) + 2(5)) = \boxed{(-3, 6)}.$$

(4) The linear map is of the form $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ for constants a, b, c, d. We use the given function values to solve for a, b, c, d

$$f(2,0) = (a(2) + b(0), c(2) + d(0)) = (2a, 2c) = (3, 1),$$
$$f(0,4) = (a(0) + b(4), c(0) + d(4)) = (4b, 4d) = (0, -1)$$

Therefore, $a = \frac{3}{2}, b = 0, c = \frac{1}{2}, d = -\frac{1}{4}$ and the linear map f is defined $f(x_1, x_2) = \left(\frac{3}{2}x_1, \frac{1}{2}x_1 - \frac{1}{4}x_2\right)$. Now, we evaluated are the desired point and get

$$f(7,1) = \left(\frac{3}{2}(7), \frac{1}{2}(7) - \frac{1}{4}(1)\right) = \left(\frac{21}{2}, \frac{7}{2} - \frac{1}{4}\right) = \boxed{\left(\frac{21}{2}, \frac{13}{4}\right)}.$$

(5) The linear map is of the form $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ for constants a, b, c, d. We use the given function values to solve for a, b, c, d

$$f(1,1) = (a(1) + b(1), c(1 + d(1)) = (a + b, c + d) = (1, 2),$$
$$f(2,3) = (a(2) + b(3), c(2) + d(3)) = (2a + 3b, 2c + 3d) = (-4, 9).$$

We now have two systems of equations:

$$\begin{cases} a + b = 1 \\ 2a + 3b = -4 \end{cases}$$

$$\begin{cases} c + d = 2 \\ 2c + 3d = 9 \end{cases}$$

We first solve the system of equations involving the constants a, b. We add -2 times equation (1) to equation (3) to get $b = -6$. Then we plug our value of b into the first equation to get $a = 7$.

Now, we solve the system of equations involving the constants c, d. We add -2 times equation (1) to the equation (3) to get $d = 5$. Then plug the value of d into the first equation to get $c = -3$. Therefore, the function f is defined

$$f(x_1, x_2) = (7x_1 - 6x_2, -3x_1 + 5x_2).$$

Evaluating the function f at the point $(-1, 5)$, we find that

$$f(-1, 5) = (7(-1) - 6(5), -3(-1) + 5(5)) = \boxed{(-37, 28).}$$

(6) The linear map is defined $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$, therefore, evaluating at $(0, 0)$, we find that

$$f(0, 0) = \boxed{(0, 0)}.$$

(7) The linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is of the form $f(x_1, x_2) = ax_1 + bx_2$. Now, we use the given data to solve for the constants a, b

$$f(1, 0) = a(1) + b(0) = a = 3,$$
$$f(0, 1) = a(0) + b(1) = b = 2.$$

We write the map $f(x_1, x_2) = 3x_1 + 2x_2$ and may now evaluate at the point $(-1, 5)$

$$f(-1, 5) = 3(-1) + 2(5) = \boxed{7}.$$

□

Problem 3. **Prove**, with an argument, or **disprove**, with a counter-example, each of the statements sentences below.

(1) If a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is such that $f(1, 0) = (1, 0)$ and $f(0, 1) = (2, 5)$. Then $f(1, 1) = (3, 5)$.

(2) If a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is such that $f(1, 0) = (1, 0)$ and $f(2, 0) = (2, 5)$. Then $f(1, 1) = (3, 5)$.

(3) If a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is such that $f(1, 3) = (1, 0)$ and $f(-2, -6) = (0, 1)$. Then $f(1, 2) = (3, 5)$.

(4) If a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is such that $f(1,0) = (1,0)$ and $f(0,1) = (2,5)$. Then $f(1,1) = (3,5)$.

(5) Any map $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form $f(x) = \alpha \cdot x$, for some $\alpha \in \mathbb{R}$ is linear.

(6) Any linear map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $f(x) = \alpha \cdot x$, for some $\alpha \in \mathbb{R}$.

(7) Any linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is of the form $f(x_1, x_2) = (\alpha_1 \cdot x_1, \alpha_2 \cdot x_2)$, for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

Solution.

(1) This statement is $\boxed{true}$.

The linear map is described $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ and using the given points we find that

$$f(1,0) = (a,c) = (1,0),$$
$$f(0,1) = (b,d) = (2,5).$$

So the linear map is defined by $f(x_1, x_2) = (x_1 + 2x_2, 5x_2)$ and evaluating at the point $(1,1)$ we have that

$$f(1,1) = (1 + 2(1), 5(1)) = (3,5).$$

(2) This statement is $\boxed{false}$.

The linear map is defined $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ and with the given points we find

$$f(1,0) = (a,c) = (1,0),$$
$$f(2,0) = (2a, 2c) = (2,5).$$

Here, we reach a contradiction because in the first equation $a = 1, c = 0$, whereas, in the second equation $a = 1, c = \frac{5}{2}$ and $0 \neq \frac{5}{2}$. Therefore, there is not a clearly defined function and we cannot determine the value of f at $(1,1)$.

Alternatively, we know that $(2,0) = 2(1,0)$ and by the linearity of f, more precisely, the scalar multiplicity condition, we should have that $f(2,0) = 2f(1,0) = 2(1,0) = (2,0) \neq (2,5)$. Therefore, there is no well-defined linear function f.

(3) This statement is $\boxed{false}$.

The linear map is given by $f(x_1, x_2) = ax_1 + bx_2, cx_1 + dx_2)$ and using the provided points we have that

$$f(1,0) = (a + 3b, c + 3d) = (1,0),$$
$$f(-2,-6) = (-2a - 6b, -2c - 6d) = (0,1).$$

We solve the first system of equations for a, b:

$$\begin{cases} a + 3b = 1 \\ -2a - 6b = 0 \end{cases}$$

We add 2 times equation (1) to equation (2) and get a contradiction of $0 = 2$. Therefore, the map is not clearly defined and there is no solution for $f(1, 2)$.

Alternatively, one might notice that the vectors $(1, 3)$ and $(-2, -6)$ are scalar multiples of one another. So under the map f the resulting vectors should also be scalar multiples of one another; however, $(1, 0)$ and $(0, 1)$ are not. In other words, using the scalar multiplication principle of linear maps we should have

$$f(-2, -6) = f(-2(1, 3)) = -2f(1, 3) = -2(1, 0) = (-2, 0) \neq (0, 1).$$

(4) This statement is $\boxed{true}$.

This is repeat of part (1), so one may find the precise linear map and check this fact as done previously.

Alternatively, one may use the linearity of the map f. Notice that the vector $(1, 1)$ is a linear combination of the vectors $(1, 0)$ and $(0, 1)$, i.e.,

$$(1, 1) = (1, 0) + (0, 1)$$

Using the linearity of the map f we know that

$$f(1, 1) = f(1, 0) + f(0, 1) = (1, 0) + (2, 5) = (3, 5).$$

(5) This statement is $\boxed{true}$.

We check the conditions of linearity, namely, vector addition and scalar multiplication.

$$f(x + y) = \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y = f(x) + f(y),$$
$$f(c \cdot x) = \alpha \cdot (c \cdot x) = (\alpha \cdot c) \cdot x = (c \cdot \alpha) \cdot x = c \cdot (\alpha \cdot x) = c \cdot f(x)$$

Therefore, the map f is linear.

(6) This statement is $\boxed{true}$.

By the properties of a linear map $f : \mathbb{R} \longrightarrow \mathbb{R}$, we have that

$$f(x) = f(x \cdot 1) = x \cdot f(1) = f(1) \cdot x.$$

Since the element 1 must map to some element in $\mathbb{R}$ call this element α, i.e., $f(1) = \alpha$, we see that

$$f(x) = f(1) \cdot x = \alpha \cdot x.$$

Therefore, any linear map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $f(x) = \alpha \cdot x$, for some $\alpha \in \mathbb{R}$.

(7) This statement is $\boxed{false}$.

A counterexample is the linear map

$$f(x_1, x_2) = (\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2, \alpha_3 \cdot x_3 + \alpha_4 \cdot x_4)$$

for some $\alpha_i \in \mathbb{R}$ where $i = \{1, 2, 3, 4\}$. Note that the map given as a counterexample is the linear map which describes all linear maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$\square$